Stochastic processes on geometric loop groups and diffeomorphism groups of real and complex manifolds, associated unitary representations.

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27 January 2001

Abstract

This article is devoted to the investigation of the Belopolskaya's and Dalecky's problem. It consists of construction of a stochastic process on an infinite dimensional Lie group G which does not satisfy locally the Campbell-Hausdorff formula and construction of a dense subgroup G' in G such that a transition measure is quasi-invariant and differentiable relative to the left or right action of G' on G. Geometric loop groups and diffeomorphism groups of Sobolev classes of smoothness are investigated for finite dimensional and also infinite dimensional real manifolds. Such groups also are defined and studied for complex manifolds finite and infinite dimensional. Stochastic processes are considered on free loop spaces, geometric loop and diffeomorphism groups of real and complex manifolds. They are used for investigations of Wiener differentiable quasi-invariant measures on such groups relative to dense subgroups. Such measures are used for the investigation of associated unitary representations of these groups.

1 Introduction.

Earlier Gaussian quasi-invariant measures on loop groups of Riemann and complex manifolds were investigated [34, 35, 37]. With the help of them irre-

ducible strongly continuous unitary representations were constructed. Gaussian measures were studied on free loop spaces also. Traditionally geometric loop groups are considered as families of mappings $f : S^1 \to N$ from the unit circle into a Riemann manifold N preserving marked points $s_0 \in S^1$ and $y_0 \in N$ under the corresponding equivalence relation caused by an action of a diffeomorphism group $Diff^\infty(S^1)$ of the circle on the free loop space [19]. But in [34, 35, 37] were defined and investigated generalized loop groups as families of mappings from one manifold $f : M \to N$ into another preserving marked points $s_0 \in M$ and $y_0 \in N$ under the corresponding equivalence relation in the free (pinned) loop space and with the help of Grothendieck construction of an Abelian group from a commutative monoid with the unit and the cancellation property with rather mild conditions on M and N for finite and infinite dimensional real and complex manifolds.

If consider the composition of two nontrivial pinned in the marked point s_0 loops of class C^n, where $n \geq 1$, then the resulting loop is continuous, but generally not of class C^n as it can be lightly seen on examples of the unit circle S^1 and the unit sphere S^2. If for the S^1 case f is a C^n-loop with $n \geq 1$, then f and f' are continuous functions by the polar coordinate $\theta \in [0, 1]$ such that $f(0) = f(1)$ and $\lim_{\theta \to 0, \theta > 0} f'(\theta) = \lim \theta \to 1, \theta < 1 f'(\theta) =: f'(0)$. There are another C^n-loops g such that $g'(0) \neq f'(0)$, but $g(0) = f(0)$. Then a loop $h(\theta) := f(2\theta)$ for each $\theta \in [0, 1/2)$ and $h(\theta) := g(2\theta - 1)$ for each $\theta \in [1/2, 1]$ is a continuous loop, but not a C^1-loop. This is generally only continuous and piecewise of class C^n and such submanifolds, restrictions on which are of class C^n, can be described as submanifolds with corners. Another reason is that $S^n \vee S^n$ is a continuous retraction of S^n, but there is not any diffeomorphism between them. Also from $S^1 \times S^n$ there is a continuous mapping on S^{n+1}, but it is not a diffeomorphism (see §2.1.4). Naturally, for a definition of the smooth composition in loop monoids and loop groups manifolds with corners (with the corresponding atlases) are used. This permits to define topological loop monoids and topological loop groups. Another two reasons of the consideration of manifolds with corners are given below.

The commutative monoid is not the free (pinned) loop space, since it is obtained from the latter by factorization. For the construction of loop groups here are used manifolds M with some mild additional conditions. When M is finite dimensional over $\mathbf{C}$ we suppose that it is compact. This condition is not very restrictive, since each locally compact space has Alexandroff (one-point) compactification (see Theorem 3.5.11 in [16]). When M is infinite

dimensional over $\mathbf{C}$ it is assumed, that M is embedded as a closed bounded subset into the corresponding Banach space X_M over $\mathbf{C}$. This is necessary that to define a group structure on a quotient space of a free loop space.

The free loop space is considered as consisting of continuous functions $f : M \to N$ which are (piecewise) holomorphic in the complex case or (piecewise) continuously differentiable in the real case on $M \setminus M'$ and preserving marked points $f(s_0) = y_0$, where M' is a closed real submanifold depending on f with a codimension $codim_{\mathbf{R}} M' = 1$, $s_0 \in M$ and $y_0 \in N$ are marked points. There are two reasons to consider such class of mappings. The first is the need to define correctly compositions of elements in the loop group (see beneath). The second is the isoperimetric inequality for holomorphic loops, which can cause the condition of a loop to be constant on a sufficiently small neighbourhood of s_0 in M, if this loop is in some small neighbourhood of w_0, where $w_0(M) := \{y_0\}$ is a constant loop (see Remark 3.2 in [22]).

In this article loop groups of different classes are considered. Classes analogous to Gevrey classes and also with the usage of Sobolev classes of $f : M \setminus M' \to N$ are considered for the construction of dense loop subgroups and quasi-invariant measures. Henceforth, we consider not only orientable manifolds M and N, but also nonorientable manifolds (apart from [35], where only orientable manifolds were considered), since for a non-orientable manifold there always exists its orientable double covering manifold (see §6.5 in [1]). Loop commutative monoids with the cancellation property are quotients of families of mappings f from M into a manifold N with $f(s_0) = y_0$ by the corresponding equivalence relation. For the definition of the equivalence relation here are not used groups of holomorphic diffeomorphisms because of strong restrictions on their structure caused by holomorphicity (see Theorems 1 and 2 in [9]). Groups are constructed from monoids with the help of A. Grothendieck procedure. These groups are commutative and non-locally compact. They does not have non-trivial local one-parameter subgroups $\{g^b : b \in (-a, a)\}$ with $a > 0$ for an element g corresponding to a class of a mapping $f : M \to N$, $f(s_0) = y_0$, when f is such that $\sup_{y \in N}[card(f^{-1}(y))] = k < \aleph_0$, since $g^{1/p}$ does not exist in the loop group for each prime integer p such that $p > k$ (see §2). Therefore, in each neighbourhood W of the unit element e there are elements which does not belong to any local one-parameter subgroup.

These groups are Abelian, non-locally compact and for them the Campbell-

Hausdorff formula is not valid (in an open local subgroup). Finite dimensional Lie groups satisfy locally the Campbell-Hausdorff formula. This is guarantied, if impose on a locally compact topological Hausdorff group G two conditions: it is a C^∞-manifold and the following mapping $(f,g) \mapsto f \circ g^{-1}$ from $G \times G$ into G is of class C^∞. But for infinite dimensional G the Campbell-Hausdorff formula does not follow from these conditions. Frequently topological Hausdorff groups satisfying these two conditions also are called Lie groups, though they can not have all properties of finite dimensional Lie groups, so that the Lie algebras for them do not play the same role as in the finite dimensional case and therefore Lie algebras are not so helpful. If G is a Lie group and its tangent space T_eG is a Banach space, then it is called a Banach-Lie group, sometimes it is undermined, that they satisfy the Campbell-Hausdorff formula locally for a Banach-Lie algebra T_eG. In some papers the Lie group terminology undermines, that it is finite dimensional. It is worthwhile to call Lie groups satisfying the Campbell-Hausdorff formula locally (in an open local subgroup) by Lie groups in the narrow sense; in the contrary case to call them by Lie groups in the broad sense.

Stochastic processes on Lie groups G were considered in [3, 7, 11]. The book [3] is devoted also to Lie algebras and to Lie groups satisfying the Campbell-Hausdorff formula, the theory of which differs drastically from the groups considered in this paper. General theorems about quasi-invariance and differentiabilty of transition measures on the Lie group G relative to a dense subgroup G' were given in [7, 11], but they permit to find G' only abstractly and when a local subgroup of G satisfies the Campbell-Hausdorff formula. For Lie groups which do not satisfy the Campbell-Hausdorff formula locally this question remained open, as was pointed out by Belopolskaya and Dalecky in Chapter 6. They have proposed in such cases to investigate concrete Lie groups that to find pairs G and G'. As it is well-known in mathematics problems of an existence of an object and a description of it are frequently called perpendicular. In each concrete case of G it its necessary to construct a stochastic process and G'. On the other hand, the groups considered in the present article do not satisfy the Campbell-Hausdorff formula.

Below path spaces, loop spaces, pinned loop groupoids, loop monoids, loop groups and diffeomorphism groups are considered not only for finite dimensional, but also for infinite dimensional manifolds. The path spaces also are called path groups, but they have the group structure neither in the usual algebraic sence nor in the usual topological group sence. Path spaces

are more or less known (see for example, [4, 41]). In this article they are mentioned mainly from the manifold point of view and in the generalized sense for the considered here classes of smoothness and manifold structures compatible with the manifold structures of loop groups.

In particular, loop and diffeomorphism groups are important for the development of the representation theory of non-locally compact groups. Their representation theory has many differences with the traditional representation theory of locally compact groups and finite dimensional Lie groups, because non-locally compact groups have not C^*-algebras associated with the Haar measures and they have not underlying Lie algebras and relations between representations of groups and underlying algebras (see also [36]).

In view of the A. Weil theorem if a topological Hausdorff group G has a quasi-invariant measure relative to the entire G, then G is locally compact. Since loop groups $(L^M N)_\xi$ are not locally compact, they can not have quasi-invariant measures relative to the entire group, but only relative to proper subgroups G' which can be chosen dense in $(L^M N)_\xi$, where an index ξ indicates on a class of smoothness. The same is true for diffeomorphism groups (besides holomorphic diffeomorphism groups of compact complex manifolds). Diffeomorphism groups of compact complex manifolds are finite dimensional Lie groups (see [28] and references therein). It is necessary to note that there are quite another groups with the same name loop groups, but they are infinite dimensional Banach-Lie groups of mappings $f : M \to H$ into a finite dimensional Lie group H with the pointwise group multiplication of mappings with values in H. The loop groups considered here are generalized geometric loop groups.

The traditional geometric loop groups and free loop spaces are important both in mathematics and in modern physical theories. Moreover, generalized geometric loop groups also can be used in the same fields of sciences and open new opportunities. In cohomology theory and physical applications stochastic processes and Wiener measures on the free loop spaces are used [2, 15, 21, 24, 25, 31, 33, 41]. In these papers were considered only particular cases of real free loop spaces and groups for finite dimensional manifolds, no any applications to the representation theory were given.

On the other hand, representation theory of non-locally compact groups is little developed apart from the case of locally compact groups. For locally compact groups theory of induced representations is well developed due to works of Frobenius, Mackey, etc. (see [6, 17] and references therein). But for

non-locally compact groups it is very little known. In particular geometric loop and diffeomoprphism groups have important applications in modern physical theories (see [23, 42] and references therein).

One of the main tools in the investigation of unitary represenations of nonlocally compact groups are quasi-invariant measures. In previous works of the author [38, 39] Gaussian quasi-invariant measures were constructed on diffeomorphism groups with some conditions on real manifolds. For example, compact manifolds without boundary were not considered, as well as infinite dimensional manifolds with boundary. In this article new Gevrey-Sobolev classes of smoothness for diffeomorphism groups of infinite dimensional real and complex manifolds are defined and investigated. This permits to define on them the Hilbert manifold structure. This in its turn simplifies the construction of stochastic processes and transition quasi-invariant measures on them. Wiener transition quasi-invariant measures are constructed below for wider classes of manifolds. Pairs of topological groups G and their dense subgroups G' are described precisely.

This work is devoted to the investigation of Wiener measures and stochastic processes on the generalized loop spaces, loop monoids, geometric loop groups and diffeomorphism groups. For the loop groups are considered both measures arising from the stochastic equations on them and aslo induced from the free loop space. Their quasi-invariance and differentiability relative to dense subgroups is investigated. Transition measures arising from stochastic processes on manifolds also are called Wiener measures. Then measures are used for the study of associated unitary regular and induced representations of dense subgroups G'.

Section 2 is devoted to the definitions of topological and manifold structures of loop groups and diffeomorphism groups and their dense subgroups. In section 3 Wiener processes and transition quasi-invariant differentiable measures are studied (see Theorem 3.3). Unitary representations of dense subgroups G' founded in sections 2 and 3 are investigated in section 5.

Section 4 is devoted to loop monoids as well as to loop groupoids, which are defined in §4.2. For the considered here classes of manifolds the generalized path space is defined in §4.4. All objects given in sections 2-4 were not considered by others authors, besides very specific particular cases of the diffeomorphism group $Diff^{\infty}(S^1)$ and loop groups for $M = S^1$ and path spaces for $M = [0, 1]$ outlined above. Differentiable transition Wiener measures on them are given in Theorems 4.1, 4.3 and 4.5. Basic facts and notations of

stochastic analysis on manifolds are reminded in the Appendix, that may be useful, for example, for specialists in group theory or differential geometry do not working with stochastic analysis.

2 Loop and diffeomorphism groups of real and complex finite and infinite dimensional manifolds.

To avoid misunderstandings we first give our definitions of manifolds considered here and then of loop and diffeomorphism groups. In §2.1.1 uniform atlases are defined. They are necessary on Lie groups for the construction of stochastic processes on them. In §§2.1.2-2.1.5 loop groups and in §§2.2-2.4 diffeomorphism groups are defined. In §§2.1.6-2.1.8, 2.5-2.9 necessary statements about their structures as Lie groups and manifolds are given.

For loop groups and diffeomorphism groups manifolds are supposed to be satisfying the corresponding specific conditions. They are related mainly with foliations in infinite dimensional manifolds. In the case of loop groups they are also related with a structure of manifolds with corners (see the reasons in the introduction). They are defined with the help of quadrants.

2.1.1. Remark. An atlas $At(M) = \{(U_j, \phi_j) : j\}$ of a manifold M on a Banach space X over $\mathbf{R}$ is called uniform, if its charts satisfy the following conditions:
$(U1)$ for each $x \in G$ there exist neighbourhoods $U^2_x \subset U^1_x \subset U_j$ such that for each $y \in U^2_x$ there is the inclusion $U^2_x \subset U^1_y$;
$(U2)$ the image $\phi_j(U^2_x) \subset X$ contains a ball of the fixed positive radius $\phi_j(U^2_x) \supset B(X, 0, r) := \{y : y \in X, \|y\| \le r\}$;
$(U3)$ for each pair of intersecting charts (U_1, ϕ_1) and (U_2, ϕ_2) connecting mappings $F_{\phi_2,\phi_1} = \phi_2 \circ \phi_1^{-1}$ are such that $\sup_x \|F'_{\phi_2,\phi_1}(x)\| \le C$ and $\sup_x \|F'_{\phi_1,\phi_2}(x)\| \le C$, where $C = const > 0$ does not depend on ϕ_1 and ϕ_2. For the diffeomorphism group $Diff^t_{\beta,\gamma}(M)$ and loop groups $(L^M N)_\xi$ we also suppose that manifolds satisfy conditions of [34, 35, 38, 39] such that these groups are separable, but here let M and N may be with a boundary, where
$(N1)$ N is of class not less, than (strongly) C^∞ and such that $sup_{x \in S_{j,l}} \|F^{(n)}_{\psi_j,\psi_l}(x)\| \le C_n$ for each $0 \le n \in \mathbf{Z}$, when $V_{j,l} \ne \emptyset$, $C_n > 0$ are constants, $At(N) := \{(V_j, \psi_j) : j\}$ denotes an atlas of N, $V_{j,l} := V_j \cap V_l$ are intersections of charts,

$S_{j,l} := \psi_l(V_{j,l})$, $\bigcup_j V_j = N$.

Conditions $(U1 - U3, N1)$ are supposed to be satisfied for the manifold N for loop groups, as well as for the manifold M for diffeomorphism groups. Certainly, the classes of smoothness of manifolds are supposed to be not less than that of groups.

2.1.2.1. Definition. A canonical closed subset Q of $X = \mathbf{R^n}$ or of the standard separable Hilbert space $X = l_2(\mathbf{R})$ over $\mathbf{R}$ is called a quadrant if it can be given by $Q := \{x \in X : q_j(x) \ge 0\}$, where $(q_j : j \in \Lambda_Q)$ are linearly independent elements of the topologically adjoint space X^*. Here $\Lambda_Q \subset \mathbf{N}$ (with $card(\Lambda_Q) = k \le n$ when $X = \mathbf{R^n}$) and k is called the index of Q. If $x \in Q$ and exactly j of the q_i's satisfy $q_i(x) = 0$ then x is called a corner of index j. Since the unitary space $X = \mathbf{C^n}$ or the separable Hilbert space $l_2(\mathbf{C})$ over $\mathbf{C}$ as considered over the field $\mathbf{R}$ is isomorphic with $X_{\mathbf{R}} := \mathbf{R^{2n}}$ or $l_2(\mathbf{R})$ respectively, then the above definition also describes quadrants in $\mathbf{C^n}$ and $l_2(\mathbf{C})$ in such sense. In the latter case we also consider generalized quadrants as canonical closed subsets which can be given by $Q := \{x \in X_{\mathbf{R}} : q_j(x + a_j) \ge 0, a_j \in X_{\mathbf{R}}, j \in \Lambda_Q\}$, where $\Lambda_Q \subset \mathbf{N}$ ($card(\Lambda_Q) = k \in \mathbf{N}$ when $dim_{\mathbf{R}} X_{\mathbf{R}} < \infty$).

2.1.2.2. Notation. If for each open subset $U \subset Q \subset X$ a function $f : Q \to Y$ for Banach spaces X and Y over $\mathbf{R}$ has continuous Frechét differentials $D^\alpha f|_U$ on U with $\sup_{x \in U} \|D^\alpha f(x)\|_{L(X^\alpha, Y)} < \infty$ for each $0 \le \alpha \le r$ for an integer $0 \le r$ or $r = \infty$, then f belongs to the class of smoothness $C^r(Q, Y)$, where $0 \le r \le \infty$, $L(X^k, Y)$ denotes the Banach space of bounded k-linear operators from X into Y.

2.1.2.3. Definition. A differentiable mapping $f : U \to U'$ is called a diffeomorphism if

(i) f is bijective and there exist continuous f' and $(f^{-1})'$, where U and U' are interiors of quadrants Q and Q' in X.

In the complex case we consider bounded generalized quadrants Q and Q' in $\mathbf{C^n}$ or $l_2(\mathbf{C})$ such that they are domains with piecewise C^∞-boundaries and we impose additional conditions on the diffeomorphism f:

(ii) $\bar{\partial} f = 0$ on U,

(iii) f and all its strong (Frechét) differentials (as multilinear operators) are bounded on U, where ∂f and $\bar{\partial} f$ are differential $(1,0)$ and $(0,1)$ forms respectively, $d = \partial + \bar{\partial}$ is an exterior derivative. In particular for $z = (z^1, ..., z^n) \in \mathbf{C^n}$, $z^j \in \mathbf{C}$, $z^j = x^{2j-1} + ix^{2j}$ and $x^{2j-1}, x^{2j} \in \mathbf{R}$ for each

$j = 1, ..., n$, $i = (-1)^{1/2}$, there are expressions: $\partial f := \sum_{j=1}^{n} (\partial f / \partial z^j) dz^j$, $\bar{\partial} f := \sum_{j=1}^{n} (\partial f / \partial \bar{z}^j) d\bar{z}^j$. In the infinite dimensional case there are equations: $(\partial f)(e_j) = \partial f / \partial z^j$ and $(\bar{\partial} f)(e_j) = \partial f / \partial \bar{z}^j$, where $\{e_j : j \in \mathbf{N}\}$ is the standard orthonormal base in $l_2(\mathbf{C})$, $\partial f / \partial z^j = (\partial f / \partial x^{2j-1} - i \partial f / \partial x^{2j})/2$, $\partial f / \partial \bar{z}^j = (\partial f / \partial x^{2j-1} + i \partial f / \partial x^{2j})/2$.

Cauchy-Riemann Condition (ii) means that f on U is the holomorphic mapping.

2.1.2.4. Definition and notation. A complex manifold M with corners is defined in the usual way: it is a metric separable space modelled on $X = \mathbf{C^n}$ or $X = l_2(\mathbf{C})$ and is supposed to be of class C^∞. Charts on M are denoted (U_l, u_l, Q_l), that is $u_l : U_l \to u_l(U_l) \subset Q_l$ are C^∞-diffeomorphisms, U_l are open in M, $u_l \circ {u_j}^{-1}$ are biholomorphic from domains $u_j(U_l \cap U_j) \ne \emptyset$ onto $u_l(U_l \cap U_j)$ (that is $u_j \circ u_l^{-1}$ and $u_l \circ u_j^{-1}$ are holomorphic and bijective) and $u_l \circ u_j^{-1}$ satisfy conditions $(i - iii)$ from §2.1.2.3, $\bigcup_j U_j = M$.

A point $x \in M$ is called a corner of index j if there exists a chart (U, u, Q) of M with $x \in U$ and $u(x)$ is of index $ind_M(x) = j$ in $u(U) \subset Q$. The set of all corners of index $j \ge 1$ is called the border ∂M of M, x is called an inner point of M if $ind_M(x) = 0$, so $\partial M = \bigcup_{j \ge 1} \partial^j M$, where $\partial^j M := \{x \in M : ind_M(x) = j\}$.

For the real manifold with corners on the connecting mappings $u_l \circ u_j^{-1} \in C^\infty$ of real charts is imposed only Condition 2.1.2.3(i).

2.1.2.5. Definition of a submanifold with corners. A subset $Y \subset M$ is called a submanifold with corners of M if for each $y \in Y$ there exists a chart (U, u, Q) of M centered at y (that is $u(y) = 0$) and there exists a quadrant $Q' \subset \mathbf{C^k}$ or in $l_2(\mathbf{C})$ such that $Q' \subset Q$ and $u(Y \cap U) = u(U) \cap Q'$. A submanifold with corners Y of M is called neat, if the index in Y of each $y \in Y$ coincides with its index in M.

Analogously for real manifolds with corners for $\mathbf{R^k}$ and $\mathbf{R^n}$ or $l_2(\mathbf{R})$ instead of $\mathbf{C^k}$ and $\mathbf{C^n}$ or $l_2(\mathbf{C})$.

2.1.2.6. Term a complex manifold. Henceforth, the term a complex manifold N modelled on $X = \mathbf{C^n}$ or $X = l_2(\mathbf{C})$ means a metric separable space supplied with an atlas $\{(U_j, \phi_j) : j \in \Lambda_N\}$ such that:

(i) U_j is an open subset of N for each $j \in \Lambda_N$ and $\bigcup_{j \in \Lambda_N} U_j = N$, where $\Lambda_N \subset \mathbf{N}$;

(ii) $\phi_j : U_j \to \phi_j(U_j) \subset X$ are C^∞-diffeomorphisms, where $\phi_j(U_j)$ are C^∞-domains in X;

(iii) $\phi_j \circ \phi_m^{-1}$ is a biholomorphic mapping from $\phi_m(U_m \cap U_j)$ onto $\phi_j(U_m \cap$

U_j) while $U_m \cap U_j \neq \emptyset$. When $X = l_2(\mathbf{C})$ it is supposed, that $\phi_j \circ \phi_m^{-1}$ are Frechét (strongly) C^∞-differentiable.

2.1.3.1. Remark. Let X be either the standard separable Hilbert space $l_2 = l_2(\mathbf{C})$ over the field $\mathbf{C}$ of complex numbers or $X = \mathbf{C^n}$. Let $t \in \mathbf{N_o} := \mathbf{N} \cup \{0\}$, $\mathbf{N} := \{1, 2, 3, ...\}$ and W be a domain with a continuous piecewise C^∞-boundary ∂W in $\mathbf{R^{2m}}$, $m \in \mathbf{N}$, that is W is a C^∞-manifold with corners and it is a canonical closed subset of $\mathbf{C^m}$, $cl(Int(W)) = W$, where $cl(V)$ denotes the closure of V, $Int(V)$ denotes the interior of V in the corresponding topological space. As usually $H^t(W, X)$ denotes the Sobolev space of functions $f : W \to X$ for which there exists a finite norm

$\|f\|_{H^t(W,X)} := (\sum_{|\alpha| \le t} \|D^\alpha f\|^2{}_{L^2(W,X)})^{1/2} < \infty$,

where $f(x) = (f^j(x) : j \in \mathbf{N})$, $f(x) \in l_2$, $f^j(x) \in \mathbf{C}$, $x \in W$,

$\|f\|^2{}_{L^2(W,X)} := \int_W \|f(x)\|^2{}_X \lambda(dx)$, λ is the Lebesgue measure on $\mathbf{R^{2m}}$, $\|z\|_{l_2} := (\sum_{j=1}^\infty |z^j|^2)^{1/2}$, $z = (z^j : j \in \mathbf{N}) \in l_2$, $z^j \in \mathbf{C}$. Then $H^\infty(W, X) := \bigcap_{t \in \mathbf{N}} H^t(W, X)$ is the uniform space with the uniformity given by the family of norms $\{\|f\|_{H^t(W,X)} : t \in \mathbf{N}\}$.

2.1.3.2. Sobolev spaces for manifolds. Let now M be a compact Riemann or complex C^∞-manifold with corners with a finite atlas $At(M) := \{(U_i, \phi_i, Q_i); i \in \Lambda_M\}$, where U_i are open in M, $\phi_i : U_i \to \phi_i(U_i) \subset Q_i \subset \mathbf{R^m}$ (or it is a subset in $\mathbf{C^m}$) are diffeomorphisms (in addition holomorphic respectively as in §2.1.2.3), (U_i, ϕ_i) are charts, $i \in \Lambda_M \subset \mathbf{N}$.

Let also N be a separable complex metrizable manifold with corners modelled either on $X = \mathbf{C^n}$ or on $X = l_2(\mathbf{C})$ respectively. Let (V_i, ψ_i, S_i) be charts of an atlas $At(N) := \{(V_i, \psi_i, S_i) : i \in \Lambda_N\}$ such that $\Lambda_N \subset \mathbf{N}$ and $\psi_i : V_i \to \psi_i(V_i) \subset S_i \subset X$ are diffeomorphisms, V_i are open in N, $\bigcup_{i \in \Lambda_N} V_i = N$. We denote by $H^t(M, N)$ the Sobolev space of functions $f : M \to N$ for which $f_{i,j} \in H^t(W_{i,j}, X)$ for each $j \in \Lambda_M$ and $i \in \Lambda_N$ for a domain $W_{i,j} \neq \emptyset$ of $f_{i,j}$, where $f_{i,j} := \psi_i \circ f \circ \phi_j{}^{-1}$, and $W_{i,j} = \phi_j(U_j \cap f^{-1}(V_i))$ are canonical closed subsets of $\mathbf{R^m}$ (or $\mathbf{C^m}$ respectively). The uniformity in $H^t(M, N)$ is given by the following base $\{(f, g) \in (H^t(M, N))^2 : \sum_{i \in \Lambda_N, j \in \Lambda_M} \|f_{i,j} - g_{i,j}\|^2{}_{H^t(W_{i,j},X)} < \epsilon\}$, where $\epsilon > 0$, $W_{i,j}$ are domains of $(f_{i,j} - g_{i,j})$. For $t = \infty$ as usually $H^\infty(M, N) := \bigcap_{t \in \mathbf{N}} H^t(M, N)$.

2.1.3.3. A uniform space of piecewise holomorphic mappings. For two complex manifolds M and N with corners let $\mathsf{O}_\Upsilon(M, N)$ denotes a space of continuous mappings $f : M \to N$ such that for each f there exists a partition Z_f of M with the help of a real C^∞-submanifold M'_f,

which may be with corners, such that its codimension over $\mathbf{R}$ in M is $codim_{\mathbf{R}} M'_f = 1$ and $M \setminus M'_f$ is a disjoint union of open complex submanifolds $M_{j,f}$ possibly with corners with $j = 1, 2, ...$ such that each restriction $f|_{M_{j,f}}$ is holomorphic with all its derivatives bounded on $M_{j,f}$. For a given partition Z (instead of Z_f) and the corresponding M' the latter subspace of continuous piecewise holomorphic mappings $f : M \to N$ is denoted by $\mathsf{O}(M, N; Z)$. The family $\{Z\}$ of all such partitions is denoted Υ. That is $\mathsf{O}_\Upsilon(M, N) = str - ind_\Upsilon \mathsf{O}(M, N; Z)$. Let also $\mathsf{O}(M, N)$ denotes the space of holomorphic mappings $f : M \to N$, $Diff^\infty(M)$ denotes a group of C^∞-diffeomorphisms of M and $Diff^{\mathsf{O}}_\Upsilon(M) := Hom(M) \cap \mathsf{O}_\Upsilon(M, M)$, where $Hom(M)$ is a group of homeomorphisms.

Let A and B be two complex manifolds with corners such that B is a submanifold of A. Then B is called a strong $C^r([0,1] \times A, A)$-retract (or $C^r([0,1], \mathsf{O}_\Upsilon(A, A))$-retract) of A if there exists a mapping $F : [0,1] \times A \to A$ such that $F(0, z) = z$ for each $z \in A$ and $F(1, A) = B$ and $F(x, A) \supset B$ for each $x \in [0,1] := \{y : 0 \le y \le 1, y \in \mathbf{R}\}$, $F(x, z) = z$ for each $z \in B$ and $x \in [0,1]$, where $F \in C^r([0,1] \times A, A)$ or $F \in C^r([0,1], \mathsf{O}_\Upsilon(A, A))$ respectively, $r \in [0, \infty)$, $F = F(x, z)$, $x \in [0,1]$, $z \in A$. Such F is called the retraction. In the case of $B = \{a_0\}$, $a_0 \in A$ we say that A is $C^r([0,1] \times A, A)$-contractible (or $C^r([0,1], \mathsf{O}_\Upsilon(A, A))$-contractible correspondingly). Two maps $f : A \to E$ and $h : A \to E$ are called $C^r([0,1] \times A, E)$-homotopic (or $C^r([0,1], \mathsf{O}_\Upsilon(A, E))$-homotopic) if there exists $F \in C^r([0,1] \times A, E)$ (or $F \in C^r([0,1], \mathsf{O}_\Upsilon(A, E))$ respectively) such that $F(0, z) = f(z)$ and $F(1, z) = h(z)$ for each $z \in A$, where E is also a complex manifold. Such F is called the homotopy.

Let M be a complex manifold with corners satisfying the following conditions:

(i) it is compact;

(ii) M is a union of two closed complex submanifolds A_1 and A_2 with corners, which are canonical closed subsets in M with $A_1 \cap A_2 = \partial A_1 \cap \partial A_2 =: A_3$ and a codimension over $\mathbf{R}$ of A_3 in M is $codim_{\mathbf{R}} A_3 = 1$;

(iii) a marked point s_0 is in A_3;

(iv) A_1 and A_2 are $C^0([0,1], \mathsf{O}_\Upsilon(A_j, A_j))$-contractible into a marked point $s_0 \in A_3$ by mappings $F_j(x, z)$, where either $j = 1$ or $j = 2$. There can be considered more general condition of $C^0([0,1], \mathsf{O}_\Upsilon(A_j, A_j))$-contractibility of A_j on $X_0 \cap A_j$, where X_0 is a closed subset in M, $j = 1$ or $j = 2$, $s_0 \in X_0$.

We consider all finite partitions $Z := \{M_k : k \in \Xi_Z\}$ of M such that M_k are complex submanifolds (of M), which may be with corners and $\bigcup_{k=1}^s M_k =$

M, $\Xi_Z = \{1, 2, ..., s\}$, $s \in \mathbf{N}$ depends on Z, M_k are canonical closed subsets of M. We denote by $\tilde{diam}(Z) := \sup_k(diam(M_k))$ the diameter of the partition Z, where $diam(A) = \sup_{x,y\in A} |x-y|_{\mathbf{C^n}}$ is a diameter of a subset A in $\mathbf{C^n}$, since each finite dimensional manifold M can be embedded into $\mathbf{C^n}$ with the corresponding $n \in \mathbf{N}$. We suppose also that $M_i \cap M_j \subset M'$ and $\partial M_j \subset M'$ for each $i \ne j$, where M' is a closed C^∞-submanifold (which may be with corners) in M with the codimension $codim_{\mathbf{R}}(M') = 1$ of M' in M, $M' = \bigcup_{j\in\Gamma_Z} {M'}_j$, ${M'}_j$ are C^∞-submanifolds of M, Γ_Z is a finite subset of $\mathbf{N}$.

We denote by $H^t(M, N; Z)$ a space of continuous functions $f : M \to N$ such that $f|_{(M\setminus M')} \in H^t(M \setminus M', N)$ and $f|_{[Int(M_i)\cup(M_i\cap {M'}_j)]} \in H^t(Int(M_i) \cup (M_i \cap {M'}_j), N)$, when $\partial M_i \cap {M'}_j \ne \emptyset$, $h^Z_{Z'} : H^t(M, N; Z) \to H^t(M, N; Z')$ are embeddings for each $Z \le Z'$ in Υ.

The ordering $Z \le Z'$ means that each submanifold $M_i^{Z'}$ from a partition Z' either belongs to the family $(M_j : j = 1, ..., k) = (M_j^Z : j = 1, ..., k)$ for Z or there exists j such that $M_i^{Z'} \subset M_j^Z$ and M_j^Z is a finite union of $M_l^{Z'}$ for which $M_l^{Z'} \subset M_j^Z$. Moreover, these $M_l^{Z'}$ are submanifolds (may be with corners) in M_j^Z.

Then we consider the following uniform space $H^t_p(M, N)$ that is the strict inductive limit $str - ind\{H^t(M, N; Z); h^{Z'}_Z; \Upsilon\}$ (the index p reminds about the procedure of partitions), where Υ is the directed family of all such Z, for which $\lim_\Upsilon \tilde{diam}(Z) = 0$.

2.1.4. Notes and definitions of loop monoids and loop groups. Let now s_0 be the marked point in M such that $s_0 \in A_3$ (see §2.1.3.3) and y_0 be a marked point in the manifold N.

(i). Suppose that M and N are connected.

Let $H^t_p(M, s_0; N, y_0) := \{f \in H^t(M, N) | f(s_0) = y_0\}$ denotes the closed subspace of $H^t(M, N)$ and ω_0 be its element such that $\omega_0(M) = \{y_0\}$, where $\infty \ge t \ge m+1$, $2m = dim_{\mathbf{R}} M$ such that $H^t \subset C^0$ due to the Sobolev embedding theorem. The following subspace $\{f : f \in H^\infty_p(M, s_0; N, y_0),\ \ \bar\partial f = 0\}$ is isomorphic with $\mathsf{O}_\Upsilon(M, s_0; N, y_0)$, since $f|_{(M\setminus M')} \in H^\infty(M \setminus M', N) = C^\infty(M \setminus M', N)$ and $\bar\partial f = 0$.

Let as usually $A \vee B := A \times \{b_0\} \cup \{a_0\} \times B \subset A \times B$ be the wedge sum of pointed spaces (A, a_0) and (B, b_0), where A and B are topological spaces with marked points $a_0 \in A$ and $b_0 \in B$. Then the wedge combination $g \vee f$ of two elements $f, g \in H^t_p(M, s_0; N, y_0)$ is defined on the domain $M \vee M$.

The spaces $\mathsf{O}_\Upsilon(J, A_3; N, y_0) := \{f \in \mathsf{O}_\Upsilon(J, N) : f(A_3) = \{y_0\}\}$ have

the manifold structure and have embeddings into $\mathsf{O}_\Upsilon(M,s_0;N,y_0)$ due to Condition 2.1.3.3(ii), where either $J=A_1$ or $J=A_2$. This induces the following embedding $\chi^*:\mathsf{O}_\Upsilon(M\vee M,s_0\times s_0;N,y_0)\hookrightarrow\mathsf{O}_\Upsilon(M,s_0;N,y_0)$. Considering $H^t_p(M,X_0;N,y_0)=\{f\in H^t(M,N):f(X_0)=\{y_0\}\}$ and $\mathsf{O}_\Upsilon(J,A_3\cup X_0;N,y_0)$ we get the embedding $\chi^*:\mathsf{O}_\Upsilon(M\vee M,X_0\times X_0;N,y_0)\hookrightarrow \mathsf{O}_\Upsilon(M,X_0;N,y_0)$. Therefore $g\circ f:=\chi^*(f\vee g)$ is the composition in $\mathsf{O}_\Upsilon(M,s_0;N,y_0)$.

The space $C^\infty(M,N)$ is dense in $C^0(M,N)$ and there is the inclusion $\mathsf{O}_\Upsilon(M,N)\subset H^\infty_p(M,N)$. Let $M_{\mathbf{R}}$ be the Riemann manifold generated by M considered over $\mathbf{R}$. Then $Diff^\infty_{s_0}(M_{\mathbf{R}})$ is a group of C^∞-diffeomorphisms η of $M_{\mathbf{R}}$ preserving the marked point s_0, that is $\eta(s_0)=s_0$. There exists the following equivalence relation R_{O} in $\mathsf{O}_\Upsilon(M,s_0;N,y_0)$: $fR_{\mathsf{O}}h$ if and only if there exist nets $\eta_n\in Diff^\infty_{s_0}(M_{\mathbf{R}})$, also f_n and $h_n\in H^\infty_p(M,s_0;N,y_0)$ with $\lim_n f_n=f$ and $\lim_n h_n=h$ such that $f_n(x)=h_n(\eta_n(x))$ for each $x\in M$ and $n\in\omega$, where ω is a directed set, $f,h\in\mathsf{O}_\Upsilon(M,s_0;N,y_0)$ and converegence is considered in $H^\infty_p(M,s_0;N,y_0)$. In general case we consider $Diff^\infty_{X_0}(M_{\mathbf{R}}):=\{f\in Diff^\infty(M_{\mathbf{R}}):f(X_0)=X_0\}$ and elements f,h in $\mathsf{O}_\Upsilon(M,X_0;N,y_0)$ and convergence in $H^\infty(M,X_0;N,y_0)$ we get the equivalence relation R_{O} in $\mathsf{O}_\Upsilon(M,X_0;N,y_0)$.

The quotient space $\mathsf{O}_\Upsilon(M,X_0;N,y_0)/R_{\mathsf{O}}=:(S^MN)_{\mathsf{O}}$ is called the loop monoid. It has a structure of topological Abelian monoid with the cancellation property (see [34, 35]). Applying the A. Grothendieck procedure (see below) to $(S^MN)_{\mathsf{O}}$ we get a loop group $(L^MN)_{\mathsf{O}}$. For the spaces $H^t_p(M,s_0;N,y_0)$ the corresponding equivalence relations are denoted $R_{t,H}$, the loop monoids are denoted by $(S^M_{\mathbf{R}}N)_{t,H}$, the loop groups are denoted by $(L^M_{\mathbf{R}}N)_{t,H}$. When real manifolds are considered we omit the index $\mathbf{R}$.

For a commutative monoid with the cancellation property $(S^MN)_{\mathsf{O}}$ there exists a commutative group $(L^MN)_{\mathsf{O}}$ equal to the Grothendieck group. This group algebraically is the quotient group F/B, where F is a free Abelian group generated by $(S^MN)_{\mathsf{O}}$ and B is a subgroup of F generated by elements $[f+g]-[f]-[g]$, f and $g\in(S^MN)_{\mathsf{O}}$, $[f]$ denotes an element of F corresponding to f. The natural mapping $\gamma:(S^MN)_{\mathsf{O}}\to(L^MN)_{\mathsf{O}}$ is injective. We supply F with a topology inherited from the Tychonoff product topology of $(S^MN)^{\mathbf{Z}}_{\mathsf{O}}$, where each element z of F is $z=\sum_f n_{f,z}[f]$, $n_{f,z}\in\mathbf{Z}$ for each $f\in(S^MN)_{\mathsf{O}}$, $\sum_f|n_{f,z}|<\infty$. In particular $[nf]-n[f]\in\mathsf{B}$, hence $(L^MN)_{\mathsf{O}}$ is the complete topological group and γ is the topological embedding such that $\gamma(f+g)=\gamma(f)+\gamma(g)$ for each $f,g\in(S^MN)_{\mathsf{O}}$, $\gamma(e)=e$, since $(z+B)\in\gamma(S^MN)_{\mathsf{O}}$, when $n_{f,z}\geq 0$ for each f, so in general $z=z^+-z^-$,

where $(z^+ + B)$ and $(z^- + B) \in \gamma(S^M N)_{\mathsf{O}}$.

2.1.5.1. Notes and Definitions. In view of §I.5 [27] a complex manifold M considered over $\mathbf{R}$ admits a (positive definite) Riemann metric g, since M is paracompact (see §§1.3 and 1.5 [27]). Due to Theorem IV.2.2 [27] there exists the Levi-Cività connection (with vanishing torsion) of $M_{\mathbf{R}}$. For the orientable manifold M suppose ν is a measure on M corresponding to the Riemann volume element w (m-form) $\nu(dx) = w(dx)/w(M)$. The Riemann volume element w is non-degenerate and non-negative, since M is orientable. For the nonorientable M consider $\tilde{M}$ its double covering orientable manifold and the quotient mapping $\theta_M : \tilde{M} \to M$, then the Riemann volume element w on $\tilde{M}$ produces the following measure $\nu(S) := w(\theta_M^{-1}(S))/(2w(\tilde{M}))$ for each Borel subset S in M.

The Christoffel symbols $\Gamma^k_{i,j}$ of the Levi-Cività derivation (see §1.8.12 [26]) are of class C^∞ for M. Then the equivalent uniformity in $H^t(M,N)$ for $0 \le t < \infty$ is given by the following base $\{(f,g) \in (H^t(M,N))^2 : \|(\psi_j \circ f - \psi_j \circ g)\|"_{H^t(M,X)} < \epsilon$, where $D^\alpha = \partial^{|\alpha|}/\partial(x^1)^{\alpha^1}...\partial(x^{2m})^{\alpha^{2m}}$, $\epsilon > 0$, $\|(\psi_j \circ f - \psi_j \circ g)\|"^2_{H^t(M,X)} := \sum_{|\alpha|\le t} \int_M |D^\alpha(\psi_j \circ f(x) - \psi_j \circ g(x))|^2 \nu(dx)\}$, $j \in \Lambda_N$, X is the Hilbert space over $\mathbf{C}$ either $\mathbf{C^n}$ or $l_2(\mathbf{C})$, x are local normal coordinates in $M_{\mathbf{R}}$. We consider submanifolds $M_{i,k}$ and $M'_{j,k}$ for each partition Z_k as in §2.1.3.3 (with Z_k instead of Z), $i \in \Xi_{Z_k}$, $j \in \Gamma_{Z_k}$, where Ξ_{Z_k} and Γ_{Z_k} are finite subsets of $\mathbf{N}$. We supply $H^\gamma(M,X;Z_k)$ with the following metric $\rho_{k,\gamma}(y) := [\sum_{i\in\Xi} \|y|_{M_{i,k}}\|"^2_{\gamma,i,k}]^{1/2}$ for $y \in H^\gamma(M,X;Z_k)$ and $\rho_{k,\gamma}(y) = +\infty$ in the contrary case, where $\Xi = \Xi_{Z_k}$, $\infty > t \ge \gamma \in \mathbf{N}$, $\gamma \ge m+1$, $\|y|_{M_{i,k}}\|"_{\gamma,i,k}$ is given analogously to $\|y\|"_{H^\gamma(M,X)}$, but with $\int_{M_{i,k}}$ instead of $\int_M$.

Let $Z^\gamma(M,X)$ be the completion of $str-ind\{H^\gamma(M,X;Z_j); h^{Z_i}_{Z_j}; \mathbf{N}\} =: Q$ relative to the following norm $\|y\|'_\gamma := \inf_k \rho_{k,\gamma}(y)$, as usually $Z^\infty(M,X) = \bigcap_{\gamma\in\mathbf{N}} Z^\gamma(M,X)$. Let $\bar{Y}^\infty(M,X) := \{f : f \in Z^\infty(M,X)$, $\bar{\partial} f_j|_{M_{j,k}} = 0$ for each $k\}$, where $f \in Z^\infty(M,X)$ imples $f = \sum_j f_j$ with $f_j \in H^\infty(M,X;Z_j)$ for each $j \in \mathbf{N}$.

For a domain W in $\mathbf{C^m}$, which is a complex manifold with corners, let $Y^{\Upsilon,a}(W,X)$ (and $Z^{\Upsilon,a}(W,X)$) be a subspace of those $f \in \bar{Y}^\infty(W,X)$ (or $f \in Z^\infty(W,X)$ respectively) for which

$$\|f\|_{\Upsilon,a} := (\sum_{j=0}^\infty (\|f\|^*{}_j)^2/[(j!)^{a_1} j^{a_2}])^{1/2} < \infty,$$

where $(\|f\|^*{}_j)^2 := (\|f\|'{}_j)^2 - (\|f\|'{}_{j-1})^2$ for $j \ge 1$ and $\|f\|^*{}_0 = \|f\|'{}_0$, $a = (a_1, a_2)$, a_1 and $a_2 \in \mathbf{R}$, $a < a'$ if either $a_1 < a'{}_1$ or $a_1 = a'{}_1$ and $a_2 < a'{}_2$.

Using the atlases $At(M)$ and $At(N)$ for M and N of class of smoothness $Y^{\Upsilon,b} \cap C^\infty$ with $a \ge b$ we get the uniform space $Y^{\Upsilon,a}(M, X_0; N, y_0)$ (and also $Z^{\Upsilon,a}(M, X_0; N, y_0)$) of mappings $f : M \to N$ with $f(X_0) = y_0$ such that $\psi_j \circ f \in Y^{\Upsilon,a}(M, X)$ (or $\psi_j \circ f \in Z^{\Upsilon,a}(M, X)$ respectively) for each j, where $\sum_{p\in\Lambda_M, j\in\Lambda_N} \|f_{p,j} - (w_0)_{p,j}\|^2_{Y^{\Upsilon,a}(W_{p,j},X)} < \infty$ for each $f \in Y^{\Upsilon,a}(M, X_0; N, y_0)$ is satisfied with $w_0(M) = \{y_0\}$, since M is compact. Substituting w_0 on a fixed mapping $\theta : M \to N$ we get the uniform space $Y^{\Upsilon,a,\theta}(M, N)$. To each equivalence class $\{g : gR_{\mathsf{O}}f\} =:< f >_{\mathsf{O}}$ there corresponds an equivalence class $< f >_{\Upsilon,a}:= cl(< f >_{\mathsf{O}} \cap Y^{\Upsilon,a}(M, X_0; N, y_0))$ (or $< f >^{\mathbf{R}}_{\Upsilon,a}:= cl(< f >_{\infty,H} \cap Z^{\Upsilon,a}(M, X_0; N, y_0)))$, where the closure is taken in $Y^{\Upsilon,a}(M, X_0; N, y_0)$ (or $Z^{\Upsilon,a}(M, X_0; N, y_0)$ respectively). This generates equivalence relations $R_{\Upsilon,a}$ and $R^{\mathbf{R}}_{\Upsilon,a}$ respectively. We denote the quotient spaces $Y^{\Upsilon,a}(M, X_0; N, y_0)/R_{\Upsilon,a}$ and $Z^{\Upsilon,a}(M, X_0; N, y_0)/R^{\mathbf{R}}_{\Upsilon,a}$ by $(S^M N)_{\Upsilon,a}$ and $(S^M_{\mathbf{R}} N)_{\Upsilon,a}$ correspondingly. Using the A. Grothendieck construction we get the loop groups $(L^M N)_{\Upsilon,a}$ and $(L^M_{\mathbf{R}} N)_{\Upsilon,a}$ respectively.

2.1.5.2. Gevrey-Sobolev classes of smoothness of loop monoids and loop groups. Notes and definitions. Let M be an infinite dimensional complex $Y^{\xi'}$-manifold with corners modelled on $l_2(\mathbf{C})$ such that

(i) there is the sequence of the canonically embedded complex submanifolds $\eta^{m+1}_m : M_m \hookrightarrow M_{m+1}$ for each $m \in \mathbf{N}$ and to $s_{0,m}$ in M_m it corresponds $s_{0,m+1} = \eta^{m+1}_m(s_{0,m})$ in M_{m+1}, $dim_{\mathbf{C}} M_m = n(m)$, $0 < n(m) < n(m+1)$ for each $m \in \mathbf{N}$, $\bigcup_m M_m$ is dense in M;

(ii) M and $At(M)$ are foliated, that is,

(α) $u_i \circ u_j^{-1}|_{u_j(U_i\cap U_j)} \to l_2$ are of the form: $u_i \circ u_j^{-1}((z^l : l \in \mathbf{N})) = (\alpha_{i,j,m}(z^1, ..., z^{n(m)}), \gamma_{i,j,m}(z^l : l > n(m)))$ for each m, when M is without a boundary. If $\partial M \ne \emptyset$ then

(β) for each boundary component M_0 of M and $U_j \cap M_0 \ne \emptyset$ we have $\phi_j : U_j \cap M_0 \to H_{l,Q}$, moreover, $\partial M_m \subset \partial M$ for each m, where $H_{l,Q} := \{z \in Q_j : x^{2l-1} \ge 0\}$, Q_j is a quadrant in l_2 such that $Int_{l_2} H_{l,Q} \ne \emptyset$ (the interior of $H_{l,Q}$ in l_2), $z^l = x^{2l-1} + ix^{2l}$, $x^j \in \mathbf{R}$, $z^l \in \mathbf{C}$ (see also §2.1.2.4);

(iii) M is embedded into l_2 as a bounded closed subset;

(iv) each M_m satisfies conditions 2.1.3.3$(i - iv)$ with $X_{0,m} := X_0 \cap M_m$, where X_0 is a closed subset in M.

Let W be a bounded canonical closed subset in $l_2(\mathbf{C})$ with a continuous

piecewise C^∞-boundary and H_m an increasing sequence of finite dimensional subspaces over $\mathbf{C}$, $H_m \subset H_{m+1}$ and $dim_{\mathbf{C}} H_m = n(m)$ for each $m \in \mathbf{N}$. Then there are spaces $P^\infty_{\Upsilon,a}(W,X) := str-ind_m Y^{\Upsilon,a}(W_m,X)$, where $W_m = W \cap H_m$ and X is a separable Hilbert space over $\mathbf{C}$.

Let $Y^\xi(W,X)$ be the completion of $P^\infty_{\Upsilon,a}(W,X)$ relative to the following norm

$$\|f\|_\xi := [\sum_{m=1}^\infty \|f|_{W_m}\|"^2{}_{Y^{\Upsilon,a}(W_m,X)}/[(n(m)!)^{1+c_1} n(m)^{c_2}]]^{1/2},$$

where $\|f|_{W_m}\|"^2{}_{Y^{\Upsilon,a}(W_m,X)} := \|f|_{W_m}\|^2{}_{Y^{\Upsilon,a}(W_m,X)} - \|f|_{W_{m-1}}\|^2{}_{Y^{\Upsilon,a}(W_{m-1},X)}$ for each $m > 1$ and $\|f|_{W_1}\|"{}_{Y^{\Upsilon,a}(W_1,X)} := \|f|_{W_1}\|_{Y^{\Upsilon,a}(W_1,X)}$; $c = (c_1, c_2)$, c_1 and $c_2 \in \mathbf{R}$, $c < c'$ if either $c_1 < c'_1$ or $c_1 = c'_1$ and $c_2 < c'_2$; $\xi = (\Upsilon, a, c)$. Let M and N be the $Y^{\Upsilon,a',c'}$-manifolds with $a' < a$ and $c' < c$.

If N is the finite dimensional complex $Y^{\Upsilon,a'}$-manifold, then it is also the $Y^{\Upsilon,a',c'}$-manifold. There exists the strict inductive limit of loop groups $(L^{M_m}N)_{\Upsilon,a} =: L^m$, since there are natural embeddings $L^m \hookrightarrow L^{m+1}$, such that each element $f \in Y^{\Upsilon,a}(M_m, X_{0,m}; N, y_0)$ is considered in $Y^{\Upsilon,a}(M_{m+1}, X_{0,m+1}; N, y_0)$ as independent from $(z^{n(m)+1}, ..., z^{n(m+1)-1})$ in the local normal coordinates $(z^1, ..., z^{n(m+1)})$ of M_{m+1}. We denote it $str-ind_m L^m =: (L^M N)_{\Upsilon,a}$ and also $str-ind_m Q^m =: Q^\infty_{\Upsilon,a}(N, y_0)$,

$str-ind_m Y^{\Upsilon,a}(M_m; N) =: Q^\infty_{\Upsilon,a}(N)$, where $Q^m := Y^{\Upsilon,a}(M_m, X_{0,m}; N, y_0)$. Then with the help of charts of $At(M)$ and $At(N)$ the space $Y^\xi(W,X)$ induces the uniformity τ in $Q^\infty_{\Upsilon,a}(N, y_0)$ and the completion of it relative to τ we denote by $Y^\xi(M, X_0; N, y_0)$, where $\xi = (\Upsilon, a, c)$ and $\sum_{p \in \Lambda_M, j \in \Lambda_N} \|f_{p,j} - (w_0)_{p,j}\|^2_{Y^\xi(W_{p,j},X)} < \infty$ for each $f \in Y^\xi(M, X_0; N, y_0)$ is supposed to be satisfied with $w_0(M) = \{y_0\}$, since each M_m is compact. Substituting w_0 on the fixed mapping $\theta : M \to N$ we get the uniform space $Y^{\xi,\theta}(M,N)$. Therefore, using classes of equivalent elements from $Q^\infty_{\Upsilon,a}(N, y_0)$ and their closures in $Y^\xi(M, X_0; N, y_0)$ as in §2.1.5.1 we get the corresponding loop monoids which are denoted $(S^M N)_\xi$. With the help of A. Grothendieck construction we get loop groups $(L^M N)_\xi$. Substituting spaces $Y^{\Upsilon,a}$ over $\mathbf{C}$ onto $Z^{\Upsilon,a}$ over $\mathbf{R}$ with respective modifications we get spaces $Z^{\Upsilon,a,c}(M,N)$ over $\mathbf{R}$, loop monoids $(S^M_{\mathbf{R}} N)_\xi$ and groups $(L^M_{\mathbf{R}} N)_\xi$ for the multi-index $\xi = (\Upsilon, a, c)$.

Let $exp : \tilde{T}N \to N$ be the exponential mapping, where $\tilde{T}N$ is a neighbourhood of N in TN [26].

The relation between manifolds with corners and usual manifolds is given by the following lemma.

2.1.6. Lemma. *If M is a complex manifold modelled on $X = \mathbf{C^n}$ or $X = l_2(\mathbf{C})$ with an atlas $At(M) = \{(V_j, \phi_j) : j\}$, then there exists an atlas $At'(M) = \{(U_k, u_k, Q_k) : k\}$ which refines $At(M)$, where (V_j, ϕ_j) are usual charts with diffeomorphisms $\phi_j : V_j \to \phi_j(V_j)$ such that $\phi_j(V_j)$ are C^∞-domains in $\mathbf{C^n}$ and (U_k, u_k, Q_k) are charts corresponding to quadrants Q_k in $\mathbf{C^n}$ or $l_2(\mathbf{C})$* (see [35] §2.3.1 and [34]).

Necessary data about structures of loop groups are given in Theorems 2.1.7 and 2.1.8.

2.1.7. Theorems. (1). *The space $(L^M N)_\xi =: G$ for $\xi = (\Upsilon, a)$ or $\xi = (\Upsilon, a, c)$ from §2.1.5 is the complete separable Abelian topological group. Moreover, G is the dense subgroup in $(L^M N)_{\mathbf{O}}$ for $\xi = (\Upsilon, a)$; G is non-discrete non-locally compact and locally connected.*

(2). *The space $X^\xi(M,N) := T_e(L^M N)_\xi$ is Hilbert for each $1 \le m = dim_{\mathbf{C}} M \le \infty$.*

(3.) *Let N be a complex Hilbert $Y^{\xi'}$-manifold with $a > a'$ and $c > c'$ for $\xi' = (\Upsilon, a')$ or $\xi' = (\Upsilon, a', c')$ respectively, then there exists a mapping $\tilde{E} : \tilde{T}(L^M N)_\xi \to (L^M N)_\xi$ defined by $\tilde{E}_\eta(v) = exp_{\eta(s)} \circ v_\eta$ on a neighbourhood V_η of the zero section in $T_\eta(L^M N)_\xi$ and it is a C^∞-mapping for $Y^{\xi'}$-manifold N by v onto a neighbourhood $W_\eta = W_e \circ \eta$ of $\eta \in (L^M N)_\xi$; $\tilde{E}$ is the uniform isomorphism of uniform spaces V_η and W_η, where $s \in M$, e is the unit element in G, $v \in V_\eta$, $1 \le m \le \infty$.*

(4). *$(L^M N)_\xi$ is the closed proper subgroup in $(L^M_{\mathbf{R}} N)_\xi$.*

The **proof** for the orientable manifolds follows from §2.9 [35] and [34], the case of nonorientable manifolds is analogous due to §§2.1.4 and 2.1.5. The latter case can be deduced also from Theorems 2.1.8 below and the case of orientable manifolds.

2.1.8. Theorems. *Suppose that manifolds M and N together with their covering manifolds $\tilde{M}$ and $\tilde{N}$ satisfy the conditions imposed above.*

(1). *Let N be the nonorientable manifold and $\theta_N : \tilde{N} \to N$ is the quotient mapping of its double covering manifold $\tilde{N}$. Then there exists a quotient group homomorphism $\tilde{\theta}_N : (L^M \tilde{N})_\xi \to (L^M N)_\xi$.*

(2). *Let M be the nonorientable manifold, then the quotient mapping $\theta_M : \tilde{M} \to M$ induces the group embedding $\tilde{\theta}_M : (L^M N)_\xi \hookrightarrow (L^{\tilde{M}} N)_\xi$.*

Proof. If M is the nonorientable manifold, then there exists the homomorphism h of the fundamental group $\pi_1(M, s_0)$ onto the two-element group $\mathbf{Z_2}$. For connected M the group $\pi_1(M, s_0)$ does not depend on the marked

point s_0 and it is denoted by $\pi_1(M)$. If M is the connected manifold, then it has a universal covering manifold M^* which is linearly connected and a fiber bundle with the group $\pi_1(M)$ and a projection $p: M^* \to M$. Using the homomorphism h one gets the orientable double covering $\tilde{M}$ of M such that $\tilde{M}$ is connected, if M is connected (see Proposition 5.9 [27] and Theorem 78 [49]). Moreover, for each $x \in M$ there exists a neighborhood U of x such that $\theta_M^{-1}(U)$ is the disjoint union of two diffeomorphic open subsets V_1 and V_2 in $\tilde{M}$, where $\theta_M : \tilde{M} \to M$ is the quotient mapping, $g : V_1 \to V_2$ is a diffeomorphism.

(1). For each $\tilde{f} \in Z^{\Upsilon,a,c}(M, \tilde{N})$ there exists $f = \theta_N \circ \tilde{f}$ in $Z^{\Upsilon,a,c}(M, N)$. This induces the quotient mapping $\bar{\theta}_N : Z^{\Upsilon,a,c}(M, \tilde{N}) \to Z^{\Upsilon,a,c}(M, N)$, hence it induces the quotient mapping $\bar{\theta}_N : Z^{\Upsilon,a,c}(M \vee M, \tilde{N}) \to Z^{\Upsilon,a,c}(M \vee M, N)$ such that $\bar{\theta}_N(f \vee h) = \bar{\theta}_N(f) \vee \bar{\theta}_N(g)$. Considering the equivalence relation in $Y^\xi(M, s_0; \tilde{N}, y_0)$ and then loop monoids we get the quotient homomorphism $(S^M\tilde{N})\xi \to (S^M N)_\xi$. With the help of A. Grothendieck construction it induces the loop groups quotient homomorphism.

(2). On the other hand, let M be the nonorientable manifold then the quotient mapping $\theta_M : \tilde{M} \to M$ induces the locally finite open covering $\{U_x : x \in M_0\}$ of M, where M_0 is a subset of M, such that each $\theta_M^{-1}(U_x)$ is the disjoint union of two open subsets $V_{x,1}$ and V_{x_2} in $\tilde{M}$ and there exists a diffeomorphism g_x of $V_{x,1}$ on $V_{x,2}$ of the same class of smoothness as M. This produces the closed subspace of all $\tilde{f} \in Z^{\Upsilon,a,c}(\tilde{M}, N)$ for which

(i) $\tilde{f}|_{V_{x,1}}(g_x^{-1}(y)) = \tilde{f}|_{V_{x,2}}(y)$ for each $y \in V_{x,2}$ and for each $x \in M_0$, where $M_0 = M_0^f$ and $\{U_x = U_x^f : x \in M_0\}$ may depend on f. If s_0 is a marked point in M, then one of the points $\tilde{s}_0$ of $\theta_M^{-1}(s_0)$ let be the marked point in $\tilde{M}$. Then θ_M induces the quotient mapping $\theta_M : \tilde{M} \vee \tilde{M} \to M \vee M$. If both M and $\tilde{M}$ satisfy the imposed above conditions on manifolds, then this induces the embedding $\bar{\theta}_M : Z^{\Upsilon,a,c}(M, N) \hookrightarrow Z^{\Upsilon,a,c}(\tilde{M}, N)$. The identity mapping $id(x) = x$ for each $x \in \tilde{M}$ evidently satisfy Condition (i). If $\tilde{f}$ is the diffeomorphism of $\tilde{M}$ satisfying Condition (i), then applying $\tilde{f}^{-2}$ to both sides of the equality we see, that it is satisfied for $\tilde{f}^{-1}$ with the same covering $\{U_x : M_0\}$. If $\tilde{f}$ and $\tilde{h}$ are two diffeomorphisms of $\tilde{M}$, then there exists $\{U_x^h : x \in M_0^h\}$ such that $\{\tilde{f}^{-1}(\theta_M^{-1}(U_x^h)) : x \in M_0^h\}$ is the locally finite covering of $\tilde{M}$. Two manifolds M and $\tilde{M}$ are metrizabel, consequently, paracompact (see Theorem 5.1.3 [16]). Due to paracompactness of M and $\tilde{M}$ there exists a locally finite covering $\{U_x^{h\circ f} : x \in M_0^{h\circ f}\}$ for which Condition (i) is satisfied, since

$\{U^f_x \cap \theta_M \circ \tilde{f}^{-1}(\theta^{-1}_{\tilde{M}}(U^h_z)) : x \in M^f_0, z \in M^h_0\}$ has a locally finite refinement. This means, that $\bar{\theta}_M : Diff^\infty(M_{\mathbf{R}}) \hookrightarrow Diff^\infty(\tilde{M}_{\mathbf{R}})$ is the group embedding. In a complete uniform space (X, U) for its subset Z a uniform space (Z, U_Z) is complete if and only if Z is closed in X relative to the topology induced by U (see Theorem 8.3.6 [16]). Since both groups are complete and the uniformity of $Diff^\infty(\tilde{M}_{\mathbf{R}})$ induces the uniformity in $Diff^\infty(M_{\mathbf{R}})$ equivalent to its own, then $\bar{\theta}_M(Diff^\infty(M_{\mathbf{R}}))$ is closed in $Diff^\infty(\tilde{M}_{\mathbf{R}})$. Considering the equivalence relation in $Z^{\Upsilon,a,c}(\tilde{M}, N)$ we get the loop monoids embedding $\tilde{\theta}_M : (S^M N)_\xi \hookrightarrow (S^{\tilde{M}} N)_\xi$ (see §§2.1.4 and 2.1.5). This produces with the help of A. Grothendieck construiction the loop groups embedding (respecting their topological group structures) $\tilde{\theta}_M : (L^M N)_\xi \hookrightarrow (L^{\tilde{M}} N)_\xi$.

2.2.1. Note. For the diffeomorphism group we also consider a compact complex manifold M. For noncompact complex M, satisfying conditions of §2.1.1 and $(N1)$ the diffeomorphism group is considered as consisting of diffeomorphisms f of class $Y^{\Upsilon,a,c}$ (see §2.1.5), that is, $(f_{i,j} - id_{i,j}) \in Y^{\Upsilon,a,c}(U_{i,j}, \phi_i(U_i))$ for each i, j, $U_{i,j}$ is a domain of definition of $(f_{i,j} - id_{i,j})$ and then analogously to the real case the diffeomorphism group $Diff^\xi(M)$ is defined, where $\xi = (\Upsilon, a, c)$, $a = (a_1, a_2)$, $c = (c_1, c_2)$, $a_1 \le -1$ and $c_1 \le -1$. This means that $Diff^\xi(M) := Y^{\xi,id}(M, M) \cap Hom(M)$.

For investigations of stochastic processes on diffeomorphism groups at first there are given below necessary definitions and statements on special kinds of diffeomorphism groups having Hilbert manifold structures.

2.2.2. Remarks and definitions. Let M and N be real manifolds on $\mathbf{R^n}$ or l_2 and satisfying Conditions 2.2.(i-vi) [38] or they may be also canonical closed submanifolds of that of in [38]. For a field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ let $l_{2,\delta}(\mathbf{K})$ be a Hilbert space of vectors $x = (x^j : x^j \in \mathbf{K}, j \in \mathbf{N})$ such that $\|x\|_{l_{2,\delta}} := \{\sum_{j=1}^\infty |x^j|^2 j^{2\delta}\}^{1/2} < \infty$. For $\delta = 0$ we omit it as the index. Let also U be an open subset in $\mathbf{R^m}$ and V be an open subset in $\mathbf{R^n}$ or l_2 over $\mathbf{R}$, where $0 \in U$ and $0 \in V$ with m and $n \in \mathbf{N}$. By $H^{l,\theta}_{\beta,\delta}(U, V)$ is denoted the following completion relative to the metric $q^l_{\beta,\delta}(f, g)$ of the family of all strongly infinite differentiable functions $f, g : U \to V$ with $q^l_{\beta,\delta}(f, \theta) < \infty$, where $\theta \in C^\infty(U, V)$, $0 \le l \in \mathbf{Z}$, $\beta \in \mathbf{R}$, $\infty > \delta \ge 0$, $q^l_{\beta,\delta}(f, g) := (\sum_{0 \le |\alpha| \le l} \|\bar{m}^{\alpha\delta} < x >^{\beta+|\alpha|} D^\alpha_x(f(x) - g(x))\|^2_{L^2})^{1/2}$, $L^2 := L^2(U, F)$ (for $F := \mathbf{R^n}$ or $F = l_{2,\delta} = l_{2,\delta}(\mathbf{R})$) is the standard Hilbert space of all classes of equivalent measurable functions $h : U \to F$ for which there exists $\|h\|_{L^2} := (\int_U |h(x)|^2_F \mu_m(dx)))^{1/2} < \infty$, μ_m denotes the Lebesgue measure on $\mathbf{R^m}$.

Let also M and N have finite atlases such that M be on $X_M := \mathbf{R^m}$ and N on $X_N := \mathbf{R^n}$ or $X_N := l_2$, $\theta : M \hookrightarrow N$ be a C^∞-mapping, for example, embedding. Then $H^{l,\theta}_{\beta,\delta}(M,N)$ denotes the completion of the family of all C^∞-functions $g, f : M \to N$ with $\kappa^l_{\beta,\delta}(f,\theta) < \infty$, where the metric is given by the following formula $\kappa^l_{\beta,\delta}(f,g) = (\sum_{i,j}[q^l_{\beta,\delta}(f_{i,j}, g_{i,j})]^2)^{1/2}$, where $f_{i,j} := \psi_i \circ f \circ \phi_j^{-1}$ with domains $\phi_j(U_j) \cap \phi_j(f^{-1}(V_i))$, $At(M) := \{(U_i, \phi_i) : i\}$ and $At(N) := \{(V_j, \psi_j) : j\}$ are atlases of M and N, U_i are open subsets in M and V_j are open subsets in M, $\phi_i : U_i \to X_M$ and $\psi_j : V_j \to X_N$ are homeomorphisms of U_i on $\phi_i(U_i)$ and V_j on $\psi_j(V_j)$, respectively. Hilbert spaces $H^{l,\theta}_{\beta,\delta}(U,F)$ and $H^l_{\beta,0}(TM)$ are called weighted Sobolev spaces, where $H^l_{\beta,\delta}(TM) := \{f : M \to TM : f \in H^l_{\beta,\delta}(M,TM), \pi \circ f(x) = x$ for each $x \in M\}$ with $\theta(x) = (x,0) \in T_xM$ for each $x \in M$. From the latter definition it follows, that for such f and g there exists $\lim_{R\to\infty} q^l_{\beta,\delta}(f|_{U^c_R}, g|_{U^c_R}) = 0$, when (U,ϕ) is a chart Hilbertian at infinity, U^c_R is an exterior of a ball of radius R in U with center in the fixed point x_0 relative to the distance function d_M in M induced by the Riemann metric g (see §2.2(v) [38]). For $\beta = 0$ or $\gamma = 0$ we omit β or γ respectively in the notation $Dif^t_{\beta,\gamma}(M) := H^{t,id}_{\beta,\gamma}(M,M) \cap Hom(M)$ and $H^{l,\theta}_{\beta,\gamma}$.

The uniform space $Dif^t_{\beta,\gamma}(M)$ has the group structure relative to the composition of diffeomorphisms and is called the diffeomorphism group, where $Hom(M)$ is the group of homeomorphisms of M.

Each topologically adjoint space $(H^l_\beta(TM))' =: H^{-l}_{-\beta}(TM)$ also is the Hilbert space with the standard norm in H' such that $\|\zeta\|_{H'} = \sup_{\|f\|_H=1} |\zeta(f)|$.

2.3. Diffeomorphism groups of Gevrey-Sobolev classes of smoothness. Notes and definitions. Let U and V be open subsets in the Euclidean space $\mathbf{R^k}$ with $k \in \mathbf{N}$ or in the standard separable Hilbert space l_2 over $\mathbf{R}$, $\theta : U \to V$ be a C^∞-function (infinitely strongly differentiable), $\infty > \delta \ge 0$ be a parameter. There exists the following metric space $H^{\{l\},\theta}_{\{\gamma\},\delta}(U,V)$ as the completion of a space of all functions $Q := \{f : f \in E^{\infty,\theta}_{\infty,\delta}(U,V)$, there exists $n \in \mathbf{N}$ such that $supp(f) \subset U \cap \mathbf{R^n}, d_{\{l\},\{\gamma\},\delta}(f,\theta) < \infty\}$ relative to the given below metric $d_{\{l\},\{\gamma\},\delta}$:

$$(i)\ d_{\{l\},\{\gamma\},\delta}(f,g) := \sup_{x\in U}(\sum_{n=1}^{\infty}(\bar{\rho}^l_{\gamma,n,\delta}(f,g))^2)^{1/2} < \infty$$

and

$\lim_{R\to\infty} d_{\{l\},\{\gamma\},\delta}(f|_{U^c_R}, g|_{U^c_R}) = 0$, when U is a chart Euclidean or Hilbertian correspondingly at infinity, f as an argument in $\bar{\rho}^l_{\gamma,n,\delta}$ is taken with the restriction on $U \cap \mathbf{R^n}$, that is, $f|_{U\cap\mathbf{R^n}} : U \cap \mathbf{R^n} \to f(U) \subset V$ (see also §§2.1-2.5 [38] and [39] about $E^{t,\theta}_{\beta,\gamma}$), $\bar{\rho}^l_{\gamma,n,\delta}(f, id)^2 := \omega_n^2(\kappa^l_{\gamma,\delta}(f|_{(U\cap\mathbf{R^n})}, id|_{(U\cap\mathbf{R^n})})^2 - \kappa^{l(n-1)}_{\gamma(n-1),\delta}(f|_{(U\cap\mathbf{R^{n-1}})}, id|_{(U\cap\mathbf{R^{n-1}})})^2)$ for each $n > 1$ and $\bar{\rho}^l_{\gamma,1,\delta}(f, id) := \omega_1(\kappa^l_{\gamma,\delta}(f|_{(U\cap\mathbf{R^1})}, id|_{(U\cap\mathbf{R^1})})$ with $q^l_{\gamma,\delta}$ and the corresponding terms $\kappa^l_{\gamma,\delta}$ from §2.2.2, $l = l(n) > n + 5$, $\gamma = \gamma(n)$ and $l(n+1) \ge l(n)$ for each n, $l(n) \ge [t] + sign\{t\} + [n/2] + 3$, $\gamma(n) \ge \beta + sign\{t\} + [n/2] + 7/2$, $\omega_{n+1} \ge n\omega_n \ge 1$. Moreover, $\bar{\rho}^l_{\gamma,n,\delta}(f, id)(x^{n+1}, x^{n+2}, ...) \ge 0$ is the metric by variables $x^1, ..., x^n$ in $H^l_{\gamma,\delta}(U \cap \mathbf{R^n}, V)$ for f as a function by $(x^1, ..., x^n)$ such that $\bar{\rho}^l_{\gamma,n,\delta}$ depends on parameters $(x^j : j > n)$. The index θ is omitted when $\theta = 0$. The series in (i) terminates $n \le k$, when $k \in \mathbf{N}$.

Let for M connecting mappings of charts be such that $(\phi_j \circ \phi_i^{-1} - id_{i,j}) \in H^{\{l'\}}_{\{\gamma'\},\chi}(U_{i,j}, l_2)$ for each $U_i \cap U_j \ne \emptyset$ and the Riemann metric g be of class of smoothness $H^{\{l'\}}_{\{\gamma'\},\chi}$, where subsets $U_{i,j}$ are open in $\mathbf{R^k}$ or in l_2 correspondingly domains of $\phi_j \circ \phi_i^{-1}$, $l'(n) \ge l(n) + 2$, $\gamma'(n) \ge \gamma(n)$ for each n, $\infty > \chi \ge \delta$, submanifolds $\{M_k : k = k(n), n \in \mathbf{N}\}$ are the same as in Lemma 3.2 [38]. Let N be some manifold satisfying analogous conditions as M. Then there exists the following uniform space $H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, N) := \{f \in E^{\infty,\theta}_{\infty,\delta}(M, N) | (f_{i,j} - \theta_{i,j}) \in H^{\{l\},\theta}_{\{\gamma\},\delta}(U_{i,j}, l_2)$ for each charts $\{U_i, \phi_i\}$ and $\{U_j, \phi_j\}$ with $U_i \cap U_j \ne \emptyset, \chi_{\{l\},\{\gamma\},\delta,\eta}(f, \theta) < \infty$ and $\lim_{R\to\infty} \chi_{\{l\},\{\gamma\},\delta,\eta}(f|_{M^c_R}, \theta|_{M^c_R}) = 0\}$ and there exists the corresponding diffeomorphism group $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M) := \{f : f \in Hom(M)$, f^{-1} and $f \in H^{\{l\},id}_{\{\gamma\},\delta,\eta}(M, M)\}$ with its topology given by the following left-invariant metric $\chi_{\{l\},\{\gamma\},\delta,\eta}(f, g) := \chi_{\{l\},\{\gamma\},\delta,\eta}(g^{-1}f, id)$,

$$(ii)\ \chi_{\{l\},\{\gamma\},\delta,\eta}(f, g) := (\sum_{i,j}(d_{\{l\},\{\gamma\},\delta}(f_{i,j}, g_{i,j}) i^\eta j^\eta)^2)^{1/2} < \infty,$$

$g_{i,j}(x) \in l_2$ and $f_{i,j}(x) \in l_2$, $\phi_i(U_i) \subset l_2$, $U_{i,j} = U_{i,j}(x^{n+1}, x^{n+2}, ...) \subset l_2$ are domains of $f_{i,j}$ by variables $x^1, ..., x^n$ for chosen $(x^j : j > n)$ due to existing foliations in M, $U_{i,j} \subset \mathbf{R^n} \hookrightarrow l_2$, when $(x^j : j > n)$ are fixed and $U_{i,j}$ is a domain in $\mathbf{R^n}$ by variables $(x^1, ..., x^n)$, where $\infty > \eta \ge 0$.

In particular, for the finite dimensional manifold M_n the group $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M_n)$ is isomorphic to the diffeomorphism group $Dif^l_{\gamma,\delta}(M_n)$ of the weighted Sobolev class of smoothness $H^l_{\gamma,\delta}$ with $l = l(n)$, $\gamma = \gamma(n)$, where $n = dim_{\mathbf{R}}(M_n) < \infty$.

2.4. Remarks. Let two sequences be given $\{l\} := \{l(n) : n \in \mathbf{N}\} \subset \mathbf{Z}$ and $\{\gamma\} := \{\gamma(n) : n \in \mathbf{N}\} \subset \mathbf{R}$, where M and $\{M_k : k = k(n), n \in \mathbf{N}\}$ are the same as in §§2.2 and 2.3. Then there exists the following space $H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M,TN)$. By $H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M|TN)$ it is denoted its subspace of functions $f : M \to TN$ with $\pi_N(f(x)) = \theta(x)$ for each $x \in M$, where $\pi_N : TN \to N$ is the natural projection, that is, each such f is a vector field along θ, $\theta : M \to N$ is a fixed C^∞-mapping. For $M = N$ and $\theta = id$ the metric space $H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M|TM)$ is denoted by $H^{\{l\}}_{\{\gamma\},\delta,\eta}(TM)$. Spaces $H^{\{l\},id}_{\{\gamma\},\delta,\eta}(M_k|TN)$ and $H^{\{l\}}_{\{\gamma\},\delta,\eta}(TM)$ are Banach spaces with the norms $\|f\|_{\{l\},\{\gamma\},\delta,\eta} := \chi_{\{l\},\{\gamma\},\delta,\eta}(f,f_0)$ denoted by the same symbol, where $f_0(x) = (x,0)$ and $pr_2 f_0(x) = 0$ for each $x \in M$. This definition can be spread on the case $l = l(n) < 0$, if take $\sup_{\|\tau\|=1} | <x>_m^{|\alpha|-\gamma(m)} (D^\alpha_x \tau_{i,j}, [\zeta_{i,j} - \xi_{i,j}])_{L^2(U_{i,j,m},l_{2,\delta})}|$ instead of $\| <x>_m^{\gamma(m)+|\alpha|} D^\alpha_x(\zeta_{i,j} - \xi_{i,j})(x)\|_{L^2(U_{i,j,m},l_{2,\delta})}$, where $\tau \in H^{-l}_{-\gamma}(M_k|TN)$, $<x>_m = (1 + \sum_{i=1}^m (x^i)^2)^{1/2}$, $U_{i,j,m} = U_{i,j,m}(x^{m+1}, x^{m+2}, ...)$ denotes the domain of the function $\zeta_{i,j}$ by $x^1, ..., x^m$ for chosen $(x^j : j > m)$, $\|\zeta\|_k(x)$ are functions by variables $(x^i : i > k)$. Further the traditional notation is used: $sign(\epsilon) = 1$ for $\epsilon > 0$, $sign(\epsilon) = -1$ for $\epsilon < 0$, $sign(0) = 0$, $\{t\} = t - [t] \ge 0$.

2.5. Lemma. *Let the manifold M and the spaces $E^t_{\beta,\delta}(TM)$ and $H^{\{l\}}_{\{\gamma\},\delta,\eta}(TM)$ be the same as in §§2.2-2.4 with $l(k) \ge [t] + [k/2] + 3 + sign\{t\}$, $\gamma(k) \ge \beta + [k/2] + 7/2 + sign\{t\}$. Then there exist constants $C > 0$ and $C_n > 1$ for each n such that $\|\zeta\|_{E^t_{\beta,\delta}(TM)} \le C\|\zeta\|_{\{l\},\{\gamma\},\delta,0}$ for each $\zeta \in H^{\{l\}}_{\{\gamma\},\delta,0}(TM)$, moreover, there can be chosen $\omega_n \ge C_n$, $C_{n+1} \ge k(n+1)(k(n+1)-1)...(k(n)+1)C_n$ for each n such that the following inequality be valid: $\|\xi\|_{C^{l'(k)}_{\gamma'(k)}(TM_k)} \le C_n\|\xi\|_{H^{l(k)}_{\gamma(k)}(TM_k)}$ for each $k = k(n)$, $l'(k) = l(k) - [k/2] - 1$, $\gamma'(k) = \gamma(k) - [k/2] - 1$ for each $\xi \in H^{l(k)}_{\gamma(k)}(TM_k)$.*

Proof. In view of theorems from [53] and the inequality $\int_{\mathbf{R^m}} <x>_m^{-m-1} dx < \infty$ (for $<x>_m$ taken in $\mathbf{R^m}$ with $x \in \mathbf{R^m}$) there exists the embedding $H^{l(n)}_{\gamma(n)}(TM_n) \hookrightarrow C^{l'(n)}_{\gamma'(n)}(TM_n)$ for each n, since $2([n/2]+1) \ge n+1$. Moreover, due to results of §III.6 [43] there exists a constant $C_n > 0$ for each $k = k(n)$, $n \in \mathbf{N}$ such that $\|\xi\|_{C^{l'(k)}_{\gamma'(k)}(TM_k)} \le C_n\|\xi\|_{H^{l(k)}_{\gamma(k)}(TM_k)}$ for each $\xi \in H^{l(k)}_{\gamma(k)}(TM_k)$. Then $D^\alpha f(x^1, ..., x^n, ...) - D^\alpha f(y^1, ..., y^n, ...) =$

$$\sum_{n=0}^{\infty}(D^\alpha f(y^1, ..., y^{n-1}, x^n, ...) - D^\alpha f(y^1, ..., y^n, x^{n+1}, ...))$$

for each $f \in H^{\{l\}}_{\{\gamma\},\delta}(TM)$ in local coordinates, where $f(y^1,...,y^{n-1},x^n,...) = f(x^1,x^2,...,x^n,...)$, if $n=0$; $\alpha=(\alpha^1,...,\alpha^m)$, $m \in \mathbf{N}$, $\alpha^i \in \mathbf{N_o} := \{0,1,2,...\}$. Hence for $x^n < y^n$ the following inequlity is satisfied:
$|D^\alpha f(y^1,...,y^{n-1},x^n,x^{n+1},...) - D^\alpha f(y^1,...,y^n,x^{n+1},...)|_{l_{2,\delta}} \bar{m}^{\alpha\delta} \le$

$$[\int_{\phi_j(U_j\cap M_k)\ni z:=(y^1,...,y^{n-1},z^n),x^n\le z^n\le y^n} |D^\alpha \partial f(y^1,...,y^{n-1},z^n,x^{n+1},...)/\partial z^n|_{l_{2,\delta}} dz^n]\bar{m}^{\alpha\delta} \le$$

$$C^1 \int\int_{\phi_j(U_j\cap M_{k(n+1)})\ni z:=(y^1,...,y^{n-1},z^n,z^{n+1}),x^n\le z^n\le y^n} sup_{x\in M}(\|f\|_{H^{l(k(n+8))}_{\gamma(k(n+8))}(M_{k(n+8)}|TM)}$$

$< z >^{-5/2}_{n+1}) dz^n dz^{n+1}(n+1)^{-2} \le C'\|f\|_{\{l\},\{\gamma\},\delta,0} \times (n+1)^{-2}$,
when $|\alpha| = \alpha^1+...+\alpha^m \le l(k)$, $k=k(n)\ge n$, $m\le n$, where $C^1 = const > 0$ and $C' = const > 0$ are constants not depending on n and k; x, y and $(y^1,..,y^n,x^{n+1},x^{n+2},...) \in \phi_j(U_j)$ for each $n \in \mathbf{N}$. This is possible due to local convexity of the subset $\phi_j(U_j) \subset l_2$. Therefore, $H^{\{l\}}_{\{\gamma\},\delta,0}(TM) \subset E^t_{\beta,\delta}(TM)$ and $\|f\|_{E^t_{\beta,\delta}(TM)} \le C\|f\|_{\{l\},\{\gamma\},\delta,0}$ for each $f \in H^{\{l\}}_{\{\gamma\},\delta,0}(TM)$, moreover, $C = C'\sum_{n=1}^\infty n^{-2} < \infty$, since $\sup_{x\in M}\sum_{j=1}^\infty g_j(x) \le \sum_{j=1}^\infty \sup_{x\in M} g_j(x)$ for each function $g: M \to [0,\infty)$ and $\lim_{R\to\infty}\|f|_{M^c_R}\|_{E^t_{\beta,\delta}(TM)} \le C\times\lim_{R\to\infty}\|f|_{M^c_R}\|_{\{l\},\{\gamma\},\delta,0} = 0$.

The space $E^t_{\beta,\delta}(TM)\cap H^{\{l\}}_{\{\gamma\},\delta,0}(TM)$ contains the corresponding cylindrical functions ζ, in particular with $supp(\zeta) \subset U_j \cap M_n$ for some $j \in \mathbf{N}$ and $k=k(n)$, $n \in \mathbf{N}$. The linear span of the family K over the field $\mathbf{R}$ of such functions ζ is dense in $E^t_{\beta,\delta}(TM)$ and in $H^{\{l\}}_{\{\gamma\},\delta,0}(TM)$ due to the Stone-Weierstrass theorem, consequently, $H^{\{l\}}_{\{\gamma\},\delta,0}(TM)$ is dense in $E^t_{\beta,\delta}(TM)$, since $\partial f/\partial x^{n+1} = 0$ for cylindrical functions f independent from x^{n+j} for $j>0$.

2.6.1. Note. For the diffeomorphism group $Diff^t_{\beta,\gamma}(\tilde{M})$ of a Banach manifold $\tilde{M}$ let M be a dense Hilbert submanifold in $\tilde{M}$ as in [38, 39].

2.6.2. Lemma. *Let $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ and M be the same as in §2.3 with values of parameters C_n from Lemma 2.5 for given $l(k)$, $\gamma(k)$ and $k=k(n)$ with $\omega_n = l(k(n))!C_n$, then $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ is the separable metrizable topological group dense in $Diff^t_{\beta,\delta}(\tilde{M})$.*

In the case of the complex manifold M the group $Diff^\xi(M)$ (see §§2.1.5 and 2.2.1) is the separable metrizable topological group.

Proof. Consider at first the real case. From the results of the paper [44] it follows that the uniform space $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M_k)$ is the topological

group for each finite dimensional submanifold M_k, since $l(k) > k+5$ and $dim_{\mathbf{R}} M_k = k$. The minimal algebraic group $G_0 := gr(Q)$ generated by the family $Q := \{f : f \in E^{\{l\},id}_{\{\gamma\},\delta}(U,V)$ for all possible pairs of charts U_i and U_j with $U = \phi_i(U_i)$ and $V = \phi_j(U_j)$, $supp(f) \subset U \cap \mathbf{R^n}$, $f \in Hom(M)$, $dim_{\mathbf{R}} M \ge n \in \mathbf{N}\}$ is dense in $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ and in $Diff^t_{\beta,\delta}(M)$ due to the Stone-Weierstrass theorem, since the union $\bigcup_k M_k$ is dense in M, where $supp(f) := cl\{x \in M : f(x) \ne x\}$, $cl(B)$ denotes the closure of a subset B in M. Therefore, $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ and $Diff^t_{\beta,\delta}(\tilde{M})$ are separable. It remains to verify that $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ is the topological group. For it we shall use Lemma 2.5. For $a > 0$ and $k \ge 1$ using integration by parts formula we get the following equality $\int_{-\infty}^{\infty}(a^2+x^2)^{-(k+2)/2}dx = ((k-1)/(ka^2))\int_{-\infty}^{\infty}(a^2+x^2)^{k/2}dx$, which takes into account the weight multipliers. Let $f, g \in Di^{\{l\}}_{\{\gamma\},\delta,\eta}(V)$ for an open subset $V = \phi_j(U_j) \subset l_2$ and $\chi_{\{l\},\{\gamma\},\delta,\eta}(f,id) < 1/2$ and $\chi_{\{l\},\{\gamma\},\delta,\eta}(g,id) < \infty$, then $\bar{\rho}^l_{\gamma,n,\delta}(g^{-1}\circ f, id) \le C_{l,n,\gamma,\delta}(\bar{\rho}^{4l}_{\gamma,n,\delta}(f,id) + \bar{\rho}^{4l}_{\gamma,n,\delta}(g,id))$, where $0 < C_{l,n,\gamma,\delta} \le 1$ is a constant dependent on l, n, γ and independent from f and g. For the Bell polynomials there is the following inequality $Y_n(1,...,1) \le n!e^n$ for each n and $Y_n(F/2,...,F/(n+1)) \le (2n)!e^n$ for $F^p := F_p = (n+p)_p := (n+p)...(n+2)(n+1)$ (see Chapter 5 in [50] and Theorem 2.5 in [5]). The Bell polynomials are given by the following formula $Y_n(fg_1,...,fg_n) := \sum_{\pi(n)}(n!f_k/(k_1!...k_n!))(g_1/1!)^{k_1}...(g_n/n!)^{k_n}$, where the sum is by all partitions $\pi(n)$ of the number n, this partition is denoted by $1^{k_1}2^{k_2}...n^{k_n}$ such that $k_1 + 2k_2 + ... + nk_n = n$ and k_i is a number of terms equal to i, the total number of terms in the partition is equal to $k = k(\pi) = k_1 + ... + k_n$, $f^k := f_k$ in the Blissar calculus notation. For each $n \in \mathbf{N}$, $l = l(n)$ and $\gamma = \gamma(n)$ the following inequality is satisfied: $\bar{\rho}^l_{\gamma,n,\delta}(f \circ g, id) \le Y_l(\bar{f}\bar{g}_1,...,\bar{f}\bar{g}_m)$, $\bar{\rho}^{l+1}_{\gamma,n,\delta}(f^{-1},id) \le (3/2)Y_l(Fp_1/2,...,Fp_m/(m+1))$, where $\bar{f}^m := \bar{f}_m = \bar{\rho}^m_{\gamma,\underline{n},\delta}(f,id)$ and $F^k := F_k = (n+k)_k$, $(n)_j := n(n-1)...(n-j+1)$, $p_k = -\bar{f}_{k+1}(3/2)^{k+1}$. Then $\sum_{n=1}^{\infty}(2l(k(n)))!e^{l(k(n))}b^{4l(k(n))}(l(k(n))!)[(4l(k(n)))!]^{-1} < \infty$ for each $0 < b < \infty$. Hence due the Cauchy-Schwarz-Bunyakovskii inequality and the condition $C_n^2 > C_n$ for each n we get: $f \circ g$ and $f^{-1} \in Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ for each f and $g \in Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$, moreover, the operations of composition an inversion are continuous.

The base of neighborhoods of id in $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ is countable, hence this group is metrizable, moreover, a metric can be chosen left-invariant due to Theorem 8.3 [20]. The case $Diff^{\xi}(M)$ for the complex manifold M is anal-

ogous.

2.7. Lemma. *Let $G' := Di^{\{l"\}}_{\{\gamma"\},\delta",\eta"}(M)$ be a subgroup of $G := Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ such that $m(n) > n/2$, $l"(n) = l(n) + m(n)n$, $\gamma"(n) = \max(\gamma(n) - m(n)n, 0)$ for each n, $inf - \lim_{n\to\infty} m(n)/n = c > 1/2$, $\delta" > \delta + 1/2$, $\infty > \eta" > \eta + 1/2$, $\eta \ge 0$ (see §2.3). Let also $G' := Diff^{\xi'}(M)$ be a subgroup of $G = Diff^{\xi}(M)$ with either $a'_1 < a_1$ and $c'_1 < c_1$ or $a'_1 = a_1$ and $a'_2 < a_2 - 1$ and $c'_1 = c_1$ and $c'_2 < c_2 - 1$ for the complex manifold M (see §§2.1.5 and 2.2.1). Then there exists a Hilbert-Schmidt operator of embedding $J : Y' \hookrightarrow Y$, where $Y := T_eG$ and $Y' := T_eG'$ are tangent Hilbert spaces.*

Proof. Consider at first the real case. The natural embeddings θ_k of the Hilbert spaces $H^{l(k)-m(k)k,b(k)}_{\gamma(k)+m(k)k,\delta}(M_k, \mathbf{R})$ into $H^{l(k),b(k)}_{\gamma(k)k,\delta}(M_k, l_{2,\delta+1+\epsilon})$ are Hilbert-Schmidt operators for each $k = k(n)$, $n \in \mathbf{N}$. For each chart (U_j, ϕ_j) there are linearly independent functions $x^m e_l < x >^{\zeta}_n /m! =: f_{m,l,n}(x)$, where $\{e_l : l \in \mathbf{N}\} \subset l_2$ is the standard orthonormal basis in l_2, $x^m := x_1^{m_1}...x_n^{m_n}$, $m! = m_1!...m_n!$, $< x >_n = (1 + \sum_{i=1}^n (x^i)^2)^{1/2}$, $n \in \mathbf{N}$, $\zeta(n) = \zeta \in \mathbf{R}$. The linear span over $\mathbf{R}$ of the family of all such functions $f(x)$ is dense in Y. Moreover, $D^\alpha f(x) = e_l \sum \binom{\alpha}{\beta}(D^\beta x^m/m!)\ (D^{\alpha-\beta} < x >^{\zeta}_n)$, where $D^\alpha = \partial_1^{\alpha^1}...\partial_n^{\alpha^n}$, $\partial_i = \partial/\partial x_i$, $\alpha = (\alpha^1, ..., \alpha^n)$, $\binom{\alpha}{\beta} = \prod_{i=1}^n \binom{\alpha^i}{\beta^i}$, $0 \le \alpha^i \in \mathbf{Z}$, $\lim_{n\to\infty} q^n/n! = 0$ for each $\infty > q > 0$, $\sum_{j,l,n=1}^\infty \sum_{|m|\ge m(n),m} [jln^n m_1...m_n]^{-(1+2\epsilon)} < \infty$ for each $0 < \epsilon < \min(c - 1/2, \eta" - \eta - 1/2, \delta" - \delta - 1/2)$, where $m = (m_1, ..., m_n)$, $|m| := m_1 + ... + m_n$, $0 \le m_i \in \mathbf{Z}$. Hence due to §§2.3 and 2.4 the embedding J is the Hilbert-Schmidt operator.

In the complex case we use the convergence of the series $\sum_{j=1}^\infty \sum_{n=1}^\infty (j!)^{a'_1-a_1}(n!)^{c'_1-c_1} < \infty$ and $\sum_{j=1}^\infty \sum_{n=1}^\infty j^{a'_2-a_2} n^{c'_2-c_2} < \infty$.

For the construction of Wiener processes on loop and diffeomorphism groups the existence of uniform atlases for them as manifolds is necessary, that is given by the following proposition.

2.8. Theorem. *Let the diffeomorphism group $G := Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ be the same as in §2.3 or $G := Diff^{\xi}(M)$ as in §2.2.1. Then*

(i) for each $H^{\{l\},id}_{\{\gamma\},\delta,\eta}(M, TM)$-vector field V its flow η_t

is a one-parameter subgroup of $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$, the curve $t \mapsto \eta_t$ is of class C^1, the mapping $\tilde{E}xp : \tilde{T}_eDi^{\{l\}}_{\{\gamma\},\delta,\eta}(M) \to Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$, is continuous and defined on the neighbourhood $\tilde{T}_eDi^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ of the zero section in $T_eDi^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$,

$V \mapsto \eta_1$;

$$(ii)\ T_f Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M), = \{V \in H^{\{l\},id}_{\{\gamma\},\delta,\eta}(M, TM) | \pi \circ V = f\};$$

$$(iii)\ (V, W) = \int_M g_{f(x)}(V_x, W_x)\mu(dx)$$

is a weak Riemannian structure on a Hilbert manifold $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$, where μ is a measure induced on M by ϕ_j and a Gaussian measure with zero mean value on l_2 produced by an injective self-adjoint operator $Q : l_2 \to l_2$ of trace class, $0 < \mu(M) < \infty$;

(iv) the Levi-Civita connection ∇ on M induces the Levi- Civita connection

$\hat{\nabla}$ on $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$;

$$(v)\ \tilde{E} : TDi^{\{l\}}_{\{\gamma\},\delta,\eta}(M) \to Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M) \text{ is defined by}$$

$\tilde{E}_\eta(V) = exp_{\eta(x)} \circ V_\eta$ on a neighbourhood $\bar{V}$ of the zero section in $T_\eta Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ and is a $H^{\{l\},id}_{\{\gamma\},\delta,\eta}$-mapping by V onto a neighbourhood $W_\eta = W_{id} \circ \eta$ of $\eta \in Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$; $\tilde{E}$ is the uniform isomorphism of uniform spaces $\bar{V}$ and W. Analogous statements are true for $Diff^\xi(M)$ with the class of smoothness $Y^{\xi,id}$ instead of $H^{\{l\},id}_{\{\gamma\},\delta,\eta}$.

Proof. Consider at first the real case. Then we have that $T_f H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, N) = \{g \in H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, TN) : \pi_N \circ g = f\}$, where $\pi_N : TN \to N$ is the canonical projection. Therefore, $TH^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, N) = H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, TN) = \bigcup_f T_f H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, N)$ and the following mapping $w_{exp} : T_f H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, N) \to H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, N)$, $w_{exp}(g) = exp \circ g$ gives charts for $H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}(M, N)$, since TN has an atlas of class $H^{\{l'(n)-1:n\}}_{\{\gamma'(n)+1:n\},\chi}$ In view of Theorem 5 about differential equations on Banach manifolds in §4.2 [30] a vector field V of class $H^{\{l\},\theta}_{\{\gamma\},\delta,\eta}$ on M defines a flow η_t of such class, that is $d\eta_t/dt = V \circ \eta_t$ and $\eta_0 = e$. From the proofs of Theorem 3.1 and Lemmas 3.2, 3.3 in [12] we get that η_t is a one-parameter subgroup of $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$, the curve $t \mapsto \eta_t$ is of class C^1, the map $\tilde{E}xp : T_e Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M) \to Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ defined by $V \mapsto \eta_1$ is continuous.

The curves of the form $t \mapsto \tilde{E}(tV)$ are geodesics for $V \in T_\eta Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ such that $d\tilde{E}(tV)/dt$ is the map $m \mapsto d(exp(tV(m))/dt = \gamma'_m(t)$ for each

$m \in M$, where $\gamma_m(t)$ is the geodesic on M, $\gamma_m(0) = \eta(m)$, $\gamma'_m(0) = V(m)$. Indeed, this follows from the existence of solutions of corresponding differential equations in the Hilbert space $H^{\{l\},\eta}_{\{\gamma\},\delta,\eta}(M|TM)$, then as in the proof of Theorem 9.1 [12].

From the definition of μ it follows that for each $x \in M$ there exists an open neighbourhood $Y \ni x$ such that $\mu(Y) > 0$ [52]. Since $t \ge 1$, the scalar product (iii) gives a weaker topology than the initial $H^{\{l\}}_{\{\gamma\},\delta,\eta}$.

Then the right multiplication $\alpha_h(f) = f \circ h$, $f \to f \circ h$ is of class C^∞ on $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ for each $h \in Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$. Moreover, $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ acts on itself freely from the right, hence we have the following principal vector bundle $\tilde{\pi} : TDi^{\{l\}}_{\{\gamma\},\delta,\eta}(M) \to Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ with the canonical projection $\tilde{\pi}$.

Analogously to [12, 38] we get the connection $\hat{\nabla} = \nabla \circ h$ on $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$. If ∇ is torsion-free then $\hat{\nabla}$ is also torsion-free. From this it follows that the existence of $\tilde{E}$ and $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ is the Hilbert manifold of class $H^{\{l'(n)-1:n\}}_{\{\gamma'(n)+1:n\},\chi,\eta}$, since exp for M is of class $H^{\{l'(n)-1:n\}}_{\{\gamma'(n)+1:n\},\delta}$, $f \to f \circ h$ is a C^∞ map with the derivative $\alpha_h : H^{\{l\},\eta}_{\{\gamma\},\delta,\eta}(M', TN) \to H^{\{l\},\eta}_{\{\gamma\},\delta,\eta}(M, TN)$ whilst $h \in H^{\{l\},\eta}_{\{\gamma\},\delta,\eta}(M, M')$,

(vi) $\tilde{E}_h(\hat{V}) := exp_{h(x)}(V(h(x)))$, where

(vii) $\hat{V}_h = V \circ h$, V is a vector field in M, $\hat{V}$ is a vector field in $Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$.

The proof in the complex case is analogous.

2.9. Proposition. *The loop group $G := (L^M N)_\xi$ from §2.1.5 and the diffeomorphism groups $G := Diff^t_{\beta,\gamma}(M)$ from [38] and $G := Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ from §2.3 and $G := Diff^\xi(M)$ from §2.2.1 have uniform atlases.*

Proof. In view of Theorems 3.1 and 3.3 [38] and Theorems 2.9.(1-4) [35] and Lemma 2.6 and Theorem 2.8 above the diffeomorphism groups G and the loop group $G := (L^M N)_\xi$ have uniform atlases (see §2.1) consistent with their topology, where M is the real manifold $1 \le t < \infty$, $0 \le \beta < \infty$, $0 \le \gamma \le \infty$ for the diffeomorphism group $Diff^t_{\beta,\gamma}(M)$ (see [38]). Others parameters are specified in the cited paaagraphs. They also include the particular cases of finite dimensional manifolds M and N.

The case of complex compact M for $G := Diff^\infty(M)$ is trivial, since $Diff^\infty(M)$ is the finite dimensional Lie group for such M [28].

In view of Theorems 2.1.7, 2.8 and Formulas 2.8.(vi, vii) above and Theorem 3.3 [38] that to satisfy conditions $(U1, U2)$ of §2.1.1 it is sufficient to find an atlas $At(G)$ of each such group G, for which U_1 is a neighbourhood of e, U^1_x and U^2_x are for $x = e$ such that $\phi_1(U_1)$ contains a ball of radius $r > 0$.

Due to the existence of the left-invariant metrics in each such topological groups and its paracompactness and separability we can take a locally finite covering $\{U_j : g_j^{-1}U_j \subset U_1 : j \in \mathbf{N}\}$, where $\{g_j : j \in \mathbf{N}\}$ is a countable subset of pairwise distinct elements of the group, $g_1 = e$. Using uniform continuity of $\tilde{E}$ we can satisfy $(U1, U2)$ with $r > 0$, since the manifolds M for diffeomorphism groups and N for loop groups also have uniform atlases. Choosing U_1 in addition such that $\tilde{E}$ is bounded on U_1U_1 and using left shifts $L_hg := hg$, where h and $g \in G$, $AB := \{c : c = ab, a \in A, b \in B\}$ for $A \cup B \subset G$, and Condition $(U3)$ for M and N we get, that there exist sufficiently small neighbourhoods U_1, U_e^1 and U_e^2 with $U_e^2U_e^2 \subset U_e^1$ and $U_x^1 \subset xU_e^1$, $U_x^2 \subset xU_e^2$ for each $x \in G$ such that Conditions $(U1 - U3)$ are satisfied, since uniform atlases exist on the Banach or Hilbert tangent space T_eG.

3 Differentiable transition Wiener measures on loop and diffeomorphism groups.

3.1. Definitions and Notes. Let G be a Hausdorff topological group, we denote by $\mu : Af(G, \mu) \to [0, \infty) \subset \mathbf{R}$ a σ-additive measure. Its left shifts $\mu_\phi(E) := \mu(\phi^{-1} \circ E)$ are considered for each $E \in Af(G, \mu)$, where $Af(G, \mu)$ is the completion of $Bf(G)$ by μ-null sets, $Bf(G)$ is the Borel σ-field on G, $\phi \circ E := \{\phi \circ h : h \in E\}$, $\phi \in G$. For a monoid or a groupoid G let left shifts of a measure μ be defined by the following formula: $\mu_\phi(E) := \mu(\phi \circ E)$. Then μ is called quasi-invariant if there exists a dense subgroup G' (or submonoid or subgroupoid correspondingly) such that μ_ϕ is equivalent to μ for each $\phi \in G'$. Henceforth, we assume that a quasi-invariance factor $\rho_\mu(\phi, g) = \mu_\phi(dg)/\mu(dg)$ is continuous by $(\phi, g) \in G' \times G$, $\mu(V) > 0$ for some (open) neighbourhood $V \subset G$ of the unit element $e \in G$ and $\mu(G) < \infty$.

Let (M, F) be a space M of measures on $(G, Bf(G))$ with values in $\mathbf{R}$ and G" be a dense subgroup (or submonoid or subgroupoid) in G such that a topology F on M is compatible with G", that is, $\mu \mapsto \mu_h$ is the homomorphism of (M, F) into itself for each $h \in G$". Let F be the topology of convergence for each $E \in Bf(G)$. Suppose also that G and G" are real Banach manifolds such that the tangent space T_eG" is dense in T_eG, then TG and TG" are also Banach manifolds. Let $\Xi(G")$ denotes the set of all differentiable vector fields X on G", that is, X are sections of the tangent bundle TG". We say

that the measure μ is continuously differentiable if there exists its tangent mapping $T_\phi\mu_\phi(E)(X_\phi)$ corresponding to the strong differentiability relative to the Banach structures of the manifolds $G"$ and $TG"$. Its differential we denote $D_\phi\mu_\phi(E)$, so $D_\phi\mu_\phi(E)(X_\phi)$ is the σ-additive real measure by subsets $E \in Af(G,\mu)$ for each $\phi \in G"$ and $X \in \Xi(G")$ such that $D\mu(E) : TG" \to \mathbf{R}$ is continuous for each $E \in Af(G,\mu)$, where $D_\phi\mu_\phi(E) = pr_2 \circ (T\mu)_\phi(E)$, $pr_2 : p \times \mathbf{F} \to \mathbf{F}$ is the projection in TN, $p \in N$, $T_pN = \mathbf{F}$, N is another real Banach differentiable manifold modelled on a Banach space $\mathbf{F}$, for a differentiable mapping $V : G" \to N$ by $TV : TG" \to TN$ is denoted the corresponding tangent mapping, $(T\mu)_\phi(E) := T_\phi\mu_\phi(E)$. Then by induction μ is called n times continuously differentiable if $T^{n-1}\mu$ is continuously differentiable such that $T^n\mu := T(T^{n-1}\mu)$, $(D^n\mu)_\phi(E)(X_{1,\phi},...,X_{n,\phi})$ are the σ-additive real measures by $E \in Af(G,\mu)$ for each X_1,...,$X_n \in \Xi(G")$, where $(X_j)_\phi =: X_{j,\phi}$ for each $j = 1,...,n$ and $\phi \in G"$, $D^n\mu : Af(G,\mu) \otimes \Xi(G")^n \to \mathbf{R}$.

Differentiable quasi-invariant transition measures on loop and diffeomorphism groups G relative to dense subsgroups G' are given by the following theorem, where the dense subgroups G' are described precisely.

3.2. Note. Suppose that in the either $Y^{\Upsilon,b}$-Hilbert or $Y^{\Upsilon,b,d'}$-manifold N modelled on l_2 (see §2.1) there exists a dense $Y^{\Upsilon,b'}$- or $Y^{\Upsilon,b',d"}$-Hilbert submanifold N' modelled on $l_{2,\epsilon} = l_{2,\epsilon}(\mathbf{C})$ (see §2.2.2), where

(1) $a > b > b'$ and $c > d'$ and either

(2) $\infty > \epsilon > 1/2$ and $d' \ge d"$ or

(3) $\infty > \epsilon \ge 0$ and $d' > d"$ (such that either $d'_1 > d"_1$ or $d'_1 = d"_1$ and $d'_2 > d"_2 + 1$) correspondingly.

If N is finite dimensional let $N' = N$. Evidently, each $Y^{\Upsilon,b}$-manifold is the complex C^∞-manifold. Certainly we suppose, that a class of smoothness of a manifold N' is not less than that of N and classes of smoothness of M and N are not less than that of a given loop group for it as in §2.1.5 and of G' as below. For the chosen loop group $G = (L^MN)_\xi$ let its dense subgroup $G' := (L^MN')_{\xi'}$ be the same as in Theorem 2.11 [35] or Theorem 2.6 [37] or [34] with parameters:

(*a*) $\xi' = (\Upsilon, a")$ such that $a" > b$ for $\xi = \mathsf{O}$ and the $Y^{\Upsilon,b}$-manifolds M and N and the $Y^{\Upsilon,b'}$-manifold N';

(*b*) $\xi' = (\Upsilon, a")$ such that $a > a" > b$ for $\xi = (\Upsilon, a)$;

(*c*) $\xi' = (\Upsilon, a", c")$ for $\xi = (\Upsilon, a, c)$ and $dim_{\mathbf{C}}M = \infty$ such that $b < a" < a$ and $d' < c" < c$ and either (2) $\infty > \epsilon > 1$ with $d" \le d'$ or (3) $\infty > \epsilon \ge 0$ with $d" < d'$, such that either $d'_1 > d"_1$ or $d'_1 = d"_1$ and

$d'_2 > d"_2 + 1$, where M and N are $Y^{\Upsilon,b,d'}$-manifolds, N' is the $Y^{\Upsilon,b',d"}$-manifold, $1 \le dim_{\mathbf{C}} M =: m < \infty$ in the cases $(a-b)$, where either $a_1 > a"_1$ or $a_1 = a"_1$ with $a_2 > a"_2 + 1$, analogously for c and $c"$, b and b' instead of a and $a"$. For the corresponding pair $G' := (L^M_{\mathbf{R}} N')_{\xi'}$ and $G := (L^M_{\mathbf{R}} N)_{\xi}$ let indices in $(1-3)$ and $(a-c)$ be the same with substitution of $\xi = \mathsf{O}$ on $\xi = (\infty, H)$. For real manifolds M and N in addition N' is on $l_{2,\epsilon}(\mathbf{R})$.

Then the embedding $J : T_eG' \hookrightarrow T_eG$ is the Hilbert-Schmidt operator, that follows from §§2.1 and 2.7.

For the diffeomorphism group $Diff^t_{\beta,\gamma}(\tilde{M})$ of a Banach manifold $\tilde{M}$ let M be a dense Hilbert submanifold in $\tilde{M}$ as in [38, 39].

3.3. Theorem. *Let G be either a loop group or a diffeomorphism group for real or complex separable metrizable C^∞-manifolds M and N, then there exist a Wiener process on G which induce quasi-invariant infinite differentiable measures μ relative to a dense subgroup G'.*

Proof. These topological groups also have structures of C^∞-manifolds, but they do not satisfy the Campbell-Hausdorff formula in any open local subgroup. Their manifold structures and actions of G' on G will be sufficient for the construction of desired measures. Manifolds over $\mathbf{C}$ naturally have structures of manifolds over $\mathbf{R}$ also.

We take $G = \bar{G}$ and $Y = \bar{Y}$ for each loop group $(L^M N)_\xi$ outlined in 3.2.(b, c), for each diffeomorphism group $Diff^\xi(M)$ of a complex manifold M given above, for each diffeomorphism group $G := Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ for a real manifold M, since such G has the Hilbert manifold structure (see Theorems 2.1.7 and 2.8 and also Appendix). For $\bar{G} := Diff^t_{\beta,\gamma}(\tilde{M})$ there exists a Hilbert dense submanifold M in a Banach manifold $\tilde{M}$ (see §2.6) and a subgroup $G := Di^{\{l\}}_{\{\gamma\},\delta,\eta}(M)$ dense in $\bar{G}$ and a diffeomorphism subgroup G' dense in G (see the proof of Theorem 3.10 [39] and Lemma 2.6.2 above). This G' can be chosen as in Lemma 2.7.

For the chosen loop group $G = (L^M N)_\xi$ let its dense subgroup $G' := (L^M N')_{\xi'}$ be the same as in §3.2 Cases (b, c). In case 3.2.(a) let $\bar{G} = (L^M N)_\xi$ and $G = (L^M N')_{\hat{\xi}}$ with $\hat{\xi} = (\Upsilon, \hat{a})$ such that $\hat{a} > a"$, then G' let be as in 3.2.(a).

On G' there exists a 1-parameter group $\rho : \mathbf{R} \times G' \to G'$ of diffeomorphisms of G' generated by a C^∞-vector field X_ρ on G' such that $X_\rho(p) = (d\rho(s,p)/ds)|_{s=0}$, where $\rho(s+t,p) = \rho(s, \rho(t,p))$ for each $s, t \in \mathbf{R}$, $\rho(0,p) = p$, $\rho(s,*) : G' \to G'$ is the diffeomorphism for each $s \in \mathbf{R}$ (about ρ see

§1.10.8 [26]). Then each measure μ on G and ρ produce a 1-parameter family of measures $\mu_s(W) := \mu(\rho(-s, W))$. For the construction of differentiable measures on the C^∞-manifold we shall use the following statement: if $a \in C^\infty(TG', TG)$ and $A \in C^\infty(TG', L_{1,2}(TG', TG))$ and $a_x \in T_xG$ and $A_x \in L_{1,2}(T_xG', T_xG)$ for each $x \in G'$, each derivative by $x \in G'$: $a_x^{(k)}$ and $A_x^{(k)}$ is a Hilbert-Schmidt mappings into $Y = T_eG$ for each $k \in \mathbf{N}$ and $\sup_{\eta \in G} \|A_\eta(t)A^*_\eta(t)\|^{-1} \le C$, where $C > 0$ is a constant, then the transition probability $P(\tau, x, t, W) := P\{\omega : \xi(t, \omega) = x, \xi(t, \omega) \in W\}$ is continuously stronlgy C^∞-differentiable along vector fields on G', where G' is a dense C^∞-submanifold on a space Y', where Y' is a separable real Hilbert space having embedding into Y as a dense linear subspace (see Theorem 3.3 and the Remark after it in Chapter 4 [7] as well as Theorems 4.2.1, 4.3.1 and 5.3.3 [7], Definitions 3.1 above), $W \in \mathsf{F}_t$.

Now let G be a loop or a diffeomorphism group of the corresponding manifolds over the field $\mathbf{R}$ or $\mathbf{C}$. Then G has the manifold structure. If $exp^N : \tilde{T}N \to N$ is an exponential mapping of the manifold N, then it induces the exponential C^∞-mapping $\tilde{E} : \tilde{T}(L^M N)_\xi \to (L^M N)_\xi$ defined by $\tilde{E}_\eta(v) = exp^N_\eta \circ v_\eta$ (see Theorem 2.1.7), where $\tilde{T}N$ is a neighbourhood of N in a tangent bundle TN, $\eta \in (L^M N)_\xi =: G$, W_e is a neighbourhood of e in G, $W_\eta = W_e \circ \eta$. At first this mapping is defined for classes of equivalent mappings of the loop monoid $(S^M N)_\xi$ and then on elements of the group, since $exp^N_{f(x)}$ is defined for each $x \in M$ and $f \in \eta \in (S^M N)_\xi$ (see Theorem 2.9.(3) [35] and Theorem 2.4.3 [37]). The manifolds G and G' are of class C^∞ and the exponential mappings $\tilde{E}$ and $\bar{E}$ for G and G' correspondingly are of class (strongly) C^∞. The analogous connection there exists in the diffeomorphism group of the manifold M satisfying the corresponding conditions (see Theorem 3.3 [38], §2.3 and Theorem 2.8) for which: $\tilde{E}_\eta(v) = exp_{\eta(x)} \circ v_\eta$ for each $x \in M$ and $\eta \in G$. We can choose the uniform atlases $At_u(G)$ such that Christoffel symbols Γ_η are bounded on each chart (see Proposition 2.9). This mapping $\tilde{E}$ is for G as the manifold and has not relations with its group structure such as given by the Campbell-Hausdorff formula for some Lie group, for example, finite dimensional Lie group. For the case of manifolds M and N over $\mathbf{C}$ we consider G and others appearing manifolds with their structure over $\mathbf{R}$, since $\mathbf{C} = \mathbf{R} \oplus i\mathbf{R}$ as the Banach space over $\mathbf{R}$.

Then for the manifold G there exists an Itô bundle. Consider for G an Itô field U with a principal part (a_η, A_η), where $a_\eta \in T_\eta G$ and $A_\eta \in L_{1,2}(H, T_\eta G)$

and $ker(A_\eta)=\{0\}$, $\theta: H_G \to G$ is a trivial bundle with a Hilbert fiber H and $H_G := G\times H$, $L_{1,2}(\theta,\tau_\eta)$ is an operator bundle with a fibre $L_{1,2}(H,T_\eta G)$. To satisfy conditions of quasi-invariance and differentiability of transition measures theorem we choose A also such that $\sup_{\eta\in G}\|A_\eta(t)A^*_\eta(t)\|^{-1}\le C$, where $C>0$ is a constant. If an operator B is selfadjoint, then $A^\phi_\eta B A^{\phi *}_\eta$ is also selfadjoint, where $A_\eta(t) =: A^{\phi_j}_\eta(t)$ is on a chart (U_j,ϕ_j). If μ_B is a Gaussian measure on $T_\eta G$ with the correlation operator B, then $\mu_{A^\phi_\eta B A^{\phi *}_\eta}$ is the Gaussian measure on $X_{1,\eta}$, where B is selfadjoint and $ker(B)=\{0\}$, $A_\eta: T_\eta G\to X_{1,\eta}$, $X_{1,\eta}$ is a Hilbert space. We can take initially μ_B a cylindrical measure on a Hilbert space X' such that $T_\eta G'\subset X'\subset T_\eta G$. If A_η is the Hilbert-Schmidt operator with $ker(A_\eta)=\{0\}$, then $A^\phi_\eta B A^{\phi *}_\eta$ is nondegenerate selfadjoint linear operator of trace class and such the so called Radonifying operator A^ϕ_η gives the σ-additive measure $\mu_{A^\phi_\eta B A^{\phi *}_\eta}$ in the completion $X'_{1,\eta}$ of X' with respect to the norm $\|x\|_1 := \|A_\eta x\|$ (see §II.2.4 [10], §I.1.1 [52], §II.2.4 [48]). Then using cylinder subsets we get a new Gaussian σ-additive measure on $T_\eta G$, which we denote also by $\mu_{A^\phi_\eta B A^{\phi *}_\eta}$ (see also Theorems I.6.1 and III.1.1 [29]).

If $U_j\cap U_l\ne\emptyset$, then $A^{\phi_l}_\eta(t) = f_{\phi_l,\phi_j}{}' A^{\phi_l}_\eta(t) f_{\phi_l,\phi_j}{}'^{-1}$, hence the correlation operator $A^\phi_\eta B A^{\phi *}_\eta$ is selfadjoint on each chart of G, that produces the Wiener process correctly. Therefore, we can consider a stochastic process $d\xi(t,\omega)=\tilde{E}_{\xi(t,\omega)}[a_{\xi(t,\omega)}dt + A_{\xi(t,\omega)}dw]$, where w is a Wiener process on $T_\eta G$ defined with the help of nuclear nondegenerate selfadjoint positive definite operator B. The corresponding Gaussian measures $\mu_{tA^\phi_\eta B A^{\phi *}_\eta}$ for $t>0$ (for the Wiener process) are defined on the Borel σ-algebra of $T_\eta G$ and $\mu_{tA^\phi_\eta B A^{\phi *}_\eta}$ for such Hilbert-Schmidt nondegenerate linear operators A_η with $ker(A_\eta)=\{0\}$ are σ-additive (see Theorem II.2.1 [10]). When the embedding operator $T_\eta G'\hookrightarrow T_\eta G$ is of Hilbert-Schmidt class, then there exists A_η and B such that $\mu_{tA^\phi_\eta B A^{\phi *}_\eta}$ is the quasi-invariant and C^∞-differentiable measure on $T_\eta G$ relative to shifts on vectors from $T_\eta G'$ (see Theorem 26.2 [52] using Carleman-Fredholm determinant and Chapter IV [10] and §5.3 [54]). Henceforth we impose such demand on B and A_η for each $\eta\in G'$.

Consider left shifts $L_h: G\to G$ such that $L_h\eta := h\circ\eta$. Let us take $a_e\in T_eG$, $A_e\in L_{1,2}(T_eG',T_eG)$, then we put $a_x=(DL_x)a_e$ and $A_x=(DL_x)\circ A_e$ for each $x\in G$, hence $a_x\in T_eG$ and $A_x\in L_{1,2}(H_x,(DL_x)T_eG)$, where $(DL_x)T_eG = T_xG$ and $T_eG'\subset T_eG$, $H_x := (DL_x)T_eG'$. Operators L_h are

(strongly) C^∞-differentiable diffeomorphisms of G such that $D_hL_h : T_\eta G \to T_{h\eta}G$ is correctly defined, since $D_hL_h = h_*$ is the differential of h [12, 13]. In view of the choice of G' in G each covariant derivative $\nabla_{X_1}...\nabla_{X_n}(D_hL_h)Y$ is of class $L_{n+2,2}(TG'^{n+1} \times G', TG)$ for each vector fields $X_1, ..., X_n, Y$ on G' and $h \in G'$, since for each $0 \le l \in \mathbf{Z}$ the embedding of T^lG' into T^lG is of Hilbert-Schmidt class, where $T^0G := G$ (above and in [7] mappings of trace and Hilbert-Schmidt classes were defined for linear mappings on Banach and Hilbert spaces and then for mappings on vector bundles). Take a dense subgroup G' as it was otlined above and consider left shifts L_h for $h \in G'$.

The considered here groups G are separable, hence the minimal σ-algebra generated by cylindrical subalgebras $f^{-1}(\mathsf{B}_n)$, n=1,2,..., coincides with the σ-algebra B of Borel subsets of G, where $f : G \to \mathbf{R^n}$ are continuous functions, B_n is the Borel σ-algebra of $\mathbf{R^n}$. Moreover, G is the topological Radon space (see Theorem I.1.2 and Proposition I.1.7 [10]). Let $P(t_0, \psi, t, W) := P(\{\omega : \xi(t_0, \omega) = \psi, \xi(t, \omega) \in W\})$ be the transition probability of the stochastic process ξ for $0 \le t_0 < t$, which is defined on a σ-algebra B of Borel subsets in G, $W \in \mathsf{B}$, since each measure $\mu_{A^\phi_\eta B A^{\phi *}_\eta}$ is defined on the σ-algebra of Borel subsets of $T_\eta G$ (see above). On the other hand, $S(t, \tau; gx) = gS(t, \tau; x)$ is the stochastic evolution family of operators for each $0 \le t_0 \le \tau < t$. There exists $\mu(W) := P(t_0, \psi, t, W)$ such that it is a σ-additive quasi-invariant strongly C^∞-differentiable relative to the action of G' by the left shifts L_h on μ measure on G, for example, $t_0 = 0$ and $\psi = e$ with $t_0 < t$, that is, $\mu_h(W) := \mu(h^{-1}W)$ is equivalent to μ and it is strongly infinitely differentiable by $h \in G'$.

The proof in cases $G = \bar{G}$ is thus obtained. In cases $G \subset \bar{G}$ and $G \ne \bar{G}$ the use of the standard procedure of cylinder subsets induce a Weiener process and a transition measure from G on $\bar{G}$ which is quasi-invariant and C^∞-differentiable relative to G' (see aslo [39]).

3.4. Note. This proof also shows, that μ is infinitely differentiable relative to each 1-parameter group $\rho : \mathbf{R} \times G' \to G'$ of diffeomorphisms of G' generated by a C^∞-vector field X_ρ on G'. Evidently, considering different (a, A) we see that there exist $\mathsf{c} = card(\mathbf{R})$ nonequivalent Wiener processes on G and c orthogonal quasi-invariant C^∞-differentiable measures on G relative to G' (see the Kakutani theorem in [10]).

4 Differentiable Wiener transition measures on loop monoids.

This section is the consequence of the preceding sections and contains results for loop monoids as well as for loop groupoids, which are defined in §4.2. For the considered here classes of manifolds the generalized path space is defined in §4.4. Differentiable transition Wiener measures on them are given in Theorems 4.1, 4.3 and 4.5.

4.1. Theorem. *Let $G := (S^M N)_\xi$ be a loop monoid for both real or complex manifolds M and N. Then there exists a dense submonoid $G' := (S^M N')_{\xi'}$ and a stochastic process, which generates quasi-invariant strongly C^∞-differentiable measure μ on G relative to G'.*

The proof is quite analogous to that of Theorem 3.3 with the help of definiton 3.1. Pairs (ξ, ξ') and (N, N') were given above in §3.2.

4.2. Note and definition. Let now M and N be two orientable Riemann manifolds finite or infinite dimensional. If M_m is a compact manifold and $f_{n,m} \in Y^\xi(M_m, N)$ has a rank $rank(f_{n,m}(x)) = dim_{\mathbf{R}} T_x M_m$ for each $x \in M_m$, then $f_{n,m}(M_m)$ is the Y^ξ-submanifold in N and on $f_{n,m}(M_m)$ there exists the Levi-Cività connection and the Riemann volume element $\nu_{n,m}$ as in §2.1.5.1 such that $\nu_{n,m}(f_{n,m}(M_m)) = 1$. This induces local normal coordinates in $f_{n,m}(M_m)$. In particular, if $M_m = S^1$ we get the natural parameter corresponding to the length of an acr in a curve, analogously in the multidimensional case. For each function $f \in Y^\xi(M, N)$ there exists a sequence $f_{n,m(n)}|_{M_m} \in Y^\xi(M_m, N)$ converging to f, hence there are the natural coordinates for f, which are mappings $\psi_f \in Y^\xi(B, N)$ and $h_f \in Y^\xi(f(M), N)$ with $h_f \circ \psi_f = f$, where B is the unit sphere in $\mathbf{R^m}$ or l_2 over $\mathbf{R}$ correspondingly. There exists an embedding $\xi^* : Y^\xi(M \vee M, N) \hookrightarrow Y^\xi(M, N)$ (see [34, 35] and §§2.1.4, 2.1.5 above). In combination with the choice of the natural coordinates we get the following continuous composition $g \circ f$ in $G := Y^\xi(M, s_0; N, y_0)$ such that $g \circ w_0 = g$, that supplies G with the groupoid structure with the unity. Let $G' := Y^{\xi'}(M, s_0; N', y_0)$ with $\xi' = (\Upsilon, a", c")$ for $\xi = (\Upsilon, a, c)$, where $b' < a' < a" < a$ and $d" < c' < c" < c$, N' is a $Y^{\Upsilon, b', d"}$-submanifold dense in N (see also Conditions (b, c) in §3.2). Such space $Y^\xi(M, s_0; N, y_0)$ is called the generalized pinned loop space.

4.3. Theorem. *On the groupoid G there exists a stochastic process generating a quasi-invariant continuosly C^∞-differentiable measure μ relative*

to the dense subgroupoid G' (see §4.2).

Proof. Since N is the C^∞-manifold, then for each curve $f(t,x) : \mathbf{R} \times M \to N$ of class C^∞ by t there exists $\partial^l f(t,x)/\partial t^l$ for each $l \in \mathbf{N}$, hence $T^l Y^\xi(M,N) = Y^\xi(M,T^l N)$ for each $l \in \mathbf{N}$ and $Y^\xi(M,N)$ is the C^∞-manifold with the exponential mapping $(Exp^Y_g V)(x) = exp^N_{g(x)} \circ v(g(x))$ for each $x \in M$ (see Proposition 1.2.3 and Corollary 1.6.8 [26] and [14]), where $V = v \circ g$ is the vector field on $Y^\xi(M,N)$, v is the vector field on N, $g \in Y^\xi(M,N)$. Therefore, $Exp^Y V$ is of class C^∞ by Frechét on $\tilde{T}Y^\xi(M,N)$. Then $Y^\xi(M,s_0;N,y_0)$ (see the notation in §2.1.5) is its closed C^∞-submanifold with $g(s_0) = y_0$ and for it the restriction $Exp^Y|_{\tilde{T}Y^\xi(M,s_0;N,y_0)}$ also is defined and is of class C^∞. In view of §3.2 the embedding of $Y^{\xi'}(M,N')$ into $Y^\xi(M,N)$ is of Hilbert-Schmidt class. Repeating almost the same arguments (without the use of h^{-1}) for groupoids G and G' as in Theorem 3.3 we get the proof of Theorem 4.3.

4.4. Let $Y^\xi(M,N)$ be as in §2.1.5, then $Y^\xi(M,N)$ be called the generalized path space, where a fixed mapping θ is omitted. If $M_k = [0,1]^k$ are submanifolds in M, $k = 1,2,...$, such that $\bigcup_k M_k$ is dense in M, then the subspace $Y^\xi_l(M,N) := \{f : f \in Y^\xi(M,N), f(x) = f(y) \text{ when } x^k = y^k \pmod 1 \text{ for each } k\}$ is called the loop space, where $x = (x^k : k = 1,2,...,x^k \in \mathbf{R}) \in M$. Let ξ and ξ' be the same as in §4.2.

4.5. Theorem. *On $Y^\xi(M,N)$ and $Y^\xi_l(M,N)$ there exists a Wiener process such that it generates quasi-invariant measures relative to vector fileds of $Y^{\xi'}(M,N)$ and $Y^{\xi'}_l(M,N)$, respectively.*

Proof. $Y^\xi(M,N)$, $Y^\xi_l(M,N)$, $Y^{\xi'}(M,N)$ and $Y^{\xi'}_l(M,N)$ are C^∞-manifolds with of class C^∞ exponential mappings, since the exponential mapping $\tilde{T}Y^\xi(M,N)$ generates the corresponding restriction on $\tilde{T}Y^\xi_l(M,N)$ also of class C^∞ (see the proof in §4.3). They have uniform atlases. Here we can take $a \in TG$ and $A \in L_{1,2}(\theta,\tau)$ (see also §3.3 without relations with DL_h). Each vector field X on $Y^{\xi'} =: G'$ generates the 1-parameter diffeomorphism group ρ_X of G' (see §3.4). Then repeating the major parts of the proof of §3.3 without L_h and so more simply, but using actions of vectors fields of TG' by ρ_X on $Y^\xi(M,N)$ or $Y^\xi_l(M,N)$ correspondingly we get the statement of this theorem, since $(D_X\rho_X)Y$ and $[(\nabla_X)^n(D_X\rho_X)]Y$ are of class $L_{n+2,2}((TG')^{n+2},TG)$ for each vector fields X and Y on G' and each $n \in \mathbf{N}$, where $G := Y^\xi$.

5 Unitary representations associated with quasi-invariant measures.

This section contains results for unitary representations associated with quasi-invariant measures, which may be in particular transition Wiener measures. The generalization 5.1.2 of theorems from preceding works [39, 35] is proved. It is applied in §5.2 to the considered here case of Wiener transition measures. Then applications to induced representations of nonlocally compact topological groups are given having in mind the examples of constructed quasi-invariant measures on loop groups and diffeomorphism groups.

5.1.1. Note. The transition measures $P =: \nu$ on G induce strongly continuous unitary regular representations of G' given by the following formula: $T_h^\nu f(g) := (\nu^h(dg)/\nu(dg))^{1/2} f(h^{-1}g)$ for $f \in L^2(G,\nu,\mathbf{C}) =: H$, $T_h^\nu \in U(H)$, $U(H)$ denotes the unitary group of the Hilbert space H. For the strong continuity of T_h^ν the continuity of the mapping $G' \ni h \mapsto \rho_\nu(h,g) \in L^1(G,\nu,\mathbf{C})$ and that ν is the Borel measure are sufficient, where $g \in G$, since ν is the Radon measure (see its definition in Chapter I [10]). On the other hand, the continuity of $\rho_\nu(h,g) = \nu^h(dg)/\nu(dg)$ by h from the Polish group G' into $L^1(G,\nu,\mathbf{C})$ follows from $\rho_\nu(h,g) \in L^1(G,\nu,\mathbf{C})$ for each $h \in G'$ and that G' is the topological subgroup of G. In section 3 mostly Polish groups $\bar{G}$ and G' were considered. When $\bar{G}$ was not Polish it was used an embedding into $\bar{G}$ of a Polish subgroup G such that $G' \subset G \subset \bar{G}$ and a measure on G induces a measure on $\bar{G}$ with the help of an algebra of cylindrical subsets. So the considered cases of representations reduce to the case of Polish groups (G', G).

More generally it is possible to consider instead of the group G a Polish topological space X on which G' acts jointly continuously: $\phi : (G' \times X) \ni (h,x) \mapsto hx =: \phi(h,x) \in X$, $\phi(e,x) = x$ for each $x \in X$, $\phi(v,\phi(h,x)) = \phi(vh,x)$ for each v and $h \in G'$ and each $x \in X$. If ϕ is the Borel function, then it is jointly continuous [18].

A representation $T : G' \to U(H)$ is called topologically irreducible, if there is not any unitary operator (homeomorphism) S on H and a closed (Hilbert) subspace H' in H such that H' is invariant relative to ST_hS^* for each $h \in G'$, that is, $ST_hS^*(H') \subset H'$.

A topological space S is called dense in itself if $S \subset S^d$, where S^d is the derivative set of S, that is, of all limit points $x \in cl(S \setminus \{x\})$, $x \in S$, where

$cl(A)$ denotes the closure of a subset A in S (see §1.3 [16]).

A measure ν on X is called ergodic, if for each $U \in Af(X, \nu)$ and $F \in Af(X, \nu)$ with $\nu(U) \times \nu(F) \neq 0$ there exists $h \in G'$ such that $\nu((h \circ E) \cap F) \neq 0$.

5.1.2. Theorem. *Let X be an infinite Polish topological space with a σ-additive σ-finite nonnegative nonzero ergodic Borel measure ν with $supp(\nu) = X$ and quasi-invariant relative to an infinite dense in itself Polish topological group G' acting on X by the Borel function ϕ. If*

(i) $sp_{\mathbf{C}}\{\psi| \quad \psi(g) := (\nu^h(dg)/\nu(dg))^{1/2}, h \in G'\}$ is dense in H and

(ii) for each $f_{1,j}$ and $f_{2,j}$ in H, $j = 1, ..., n$, $n \in \mathbf{N}$ and each $\epsilon > 0$ there exists $h \in G'$ such that $|(T_h f_{1,j}, f_{2,j})| \leq \epsilon |(f_{1,j}, f_{2,j})|$, when $|(f_{1,j}, f_{2,j})| > 0$. Then the regular representation $T : G' \to U(H)$ is topologically irreducible.

Proof. From Condition (i) it follows, that the vector f_0 is cyclic, where $f_0 \in H$ and $f_0(g) = 1$ for each $g \in X$. In view of $card(X) \geq \aleph_0$ and the ergodicity of ν for each $n \in \mathbf{N}$ there are subsets $U_j \in Bf(X)$ and $g_j \in G'$ such that $\nu((g_j U_j) \cap (\bigcup_{i=1,...,j-1,j+1,...,n} U_i)) = 0$ and $\prod_{j=1}^n \nu_j(U_j) > 0$. Together with Condition (ii) this implies, that there is not any finite dimensional G'-invariant subspace H' in H such that $T_h H' \subset H'$ for each $h \in G'$ and $H' \neq \{0\}$. Hence if there is a G'-invariant closed subspace $H' \neq 0$ in H it is isomorphic with the subspace $L^2(V, \nu, \mathbf{C})$, where $V \in Bf(X)$ with $\nu(V) > 0$.

Let A_G denotes a $*$-subalgebra of an algebra $\mathsf{L}(H)$ of bounded linear operators on H generated by the family of unitary operators $\{T_h : h \in G'\}$. In view of the von Neumann double commuter Theorem (see §VI.24.2 [17]) A_G" coincides with the weak and strong operator closures of A_G in $\mathsf{L}(H)$, where $\mathsf{A}_G{}'$ denotes the commuting algebra of A_G and A_G" $= (\mathsf{A}_G{}')'$.

Each Polish space is Čech-complete. By the Baire category theorem in a Čech-complete space X the union $A = \bigcup_{i=1}^\infty A_i$ of a sequence of nowhere dense subsets A_i is a codense subset (see Theorem 3.9.3 [16]). On the other hand, in view of Theorem 5.8 [20] a subgroup of a topological group is discrete if and only if it contains an isolated point. Therefore, we can choose

(i) a probability Radon measure λ on G' such that λ has not any atoms and $supp(\lambda) = G'$. In view of the strong continuity of the regular representation there exists the S. Bochner integral $\int_X T_h f(g) \nu(dg)$ for each $f \in H$, which implies its existence in the weak (B. Pettis) sence. The measures ν and λ are non-negative and bounded, hence $H \subset L^1(X, \nu, \mathbf{C})$ and $L^2(G', \lambda, \mathbf{C}) \subset L^1(G', \lambda, \mathbf{C})$ due to the Cauchy inequality. Therefore, we can apply below the Fubini Theorem (see §II.16.3 [17]). Let $f \in H$, then there exists a countable

orthonormal base $\{f^j : j \in \mathbf{N}\}$ in $H \ominus \mathbf{C}f$. Then for each $n \in \mathbf{N}$ the following set $B_n := \{q \in L^2(G', \lambda, \mathbf{C}) : (f^j, f)_H = \int_{G'} q(h)(f^j, T_h f_0)_H \lambda(dh)$ for $j = 0, ..., n\}$ is non-empty, since the vector f_0 is cyclic, where $f^0 := f$. There exists $\infty > R > \|f\|_H$ such that $B_n \cap B^R =: B_n^R$ is non-empty and weakly compact for each $n \in \mathbf{N}$, since B^R is weakly compact, where $B^R := \{q \in L^2(G', \lambda, \mathbf{C}) : \|q\| \le R\}$ (see the Alaoglu-Bourbaki Theorem in §(9.3.3) [46]). Therefore, B_n^R is a centered system of closed subsets of B^R, that is, $\cap_{n=1}^m B_n^R \ne \emptyset$ for each $m \in \mathbf{N}$, hence it has a non-empty intersection, consequently, there exists $q \in L^2(G', \lambda, \mathbf{C})$ such that

$$(ii)\ f(g) = \int_{G'} q(h) T_h f_0(g) \lambda(dh)$$

for ν-a.e. $g \in X$. If $F \in L^\infty(X, \nu, \mathbf{C})$, f_1 and $f_2 \in H$, then there exist q_1 and $q_2 \in L^2(G', \lambda, \mathbf{C})$ satisfying Equation (ii). Therefore,

$$(iii)\ (f_1, F f_2)_H =: c = \int_X \int_{G'} \int_{G'} \bar{q}_1(h_1) q_2(h_2) \rho_\nu^{1/2}(h_1, g) \rho_\nu^{1/2}(h_2, g) F(g) \lambda(dh_1) \lambda(dh_2) \nu(dg).$$

$$\text{Let } \xi(h) := \int_X \int_{G'} \int_{G'} \bar{q}_1(h_1) q_2(h_2) \rho_\nu^{1/2}(h_1, g) \rho_\nu^{1/2}(h h_2, g) \lambda(dh_1) \lambda(dh_2) \nu(dg).$$

Then there exists $\beta(h) \in L^2(G', \lambda, \mathbf{C})$ such that

(iv) $\int_{G'} \beta(h) \xi(h) \lambda(dh) = c$.

To prove this we consider two cases. If $c = 0$ it is sufficient to take β orthogonal to ξ in $L^2(G', \lambda, \mathbf{C})$. Each function $q \in L^2(G', \lambda, \mathbf{C})$ can be written as $q = q^1 - q^2 + iq^3 - iq^4$, where $q^j(h) \ge 0$ for each $h \in G'$ and $j = 1, ..., 4$, hence we obtain the corresponding decomposition for ξ, $\xi = \sum_{j,k} b^{j,k} \xi^{j,k}$, where $\xi^{j,k}$ corresponds to q_1^j and q_2^k, where $b^{j,k} \in \{1, -1, i, -i\}$. If $c \ne 0$ we can choose (j_0, k_0) for which $\xi^{j_0,k_0} \ne 0$ and

(v) β is orthogonal to others $\xi^{j,k}$ with $(j,k) \ne (j_0, k_0)$.

Otherwise, if $\xi^{j,k} = 0$ for each (j,k), then $q_l^j(h) = 0$ for each (l,j) and λ-a.e. $h \in G'$, since

$$\xi(0) = \int_X \nu(dg) \Big(\int_{G'} \bar{q}_1(h_1) \rho_\nu^{1/2}(h_1, g) \lambda(dh_1)\Big) \Big(\int_{G'} q_2(h_2) \rho_\nu^{1/2}(h_2, g) \lambda(dh_2)\Big) = 0$$

and this implies $c = 0$, which is the contradiction with the assumption $c \ne 0$. Hence there exists β satisfying conditions (iv, v).

Let $a(x) \in L^\infty(X, \nu, \mathbf{C})$, f and $g \in H$, $\beta(h) \in L^2(G', \lambda, \mathbf{C})$. Since $L^2(G', \lambda, \mathbf{C})$ is infinite dimensional, then for each finite family of $a \in \{a_1, ..., a_m\} \subset$

$L^\infty(X,\nu,\mathbf{C})$, $f\in\{f_1,...,f_m\}\subset H$ there exists $\beta(h)\in L^2(G',\lambda,\mathbf{C})$, $h\in G'$, such that β is orthogonal to $\int_X \bar f_s(g)[f_j(h^{-1}g)(\rho_\nu(h,g))^{1/2}-f_j(g)]\nu(dg)$ for each $s,j=1,...,m$. Hence each operator of multiplication on $a_j(g)$ belongs to A_G", since due to Formula (iv) and Condition (v) there exists $\beta(h)$ such that

$$(f_s,a_jf_l)=\int_X\int_{G'}\bar f_s(g)\beta(h)(\rho_\nu(h,g))^{1/2}f_l(h^{-1}g)\lambda(dh)\nu(dg)=$$

$$=\int_X\int_{G'}\bar f_s(g)\beta(h)(T_hf_l(g))\lambda(dh)\nu(dg),\ \int_X\bar f_s(g)a_j(g)f_l(g)\nu(dg)=$$

$$=\int_X\int_{G'}\bar f_s(g)\beta(h)f_l(g)\lambda(dh)\nu(dg)=(f_s,a_jf_l).$$

Hence A_G" contains subalgebra of all operators of multiplication on functions from $L^\infty(X,\nu,\mathbf{C})$. With G' and a Banach algebra A the trivial Banach bundle $\mathsf{B}=\mathsf{A}\times G'$ is associative, in particular let $\mathsf{A}=\mathbf{C}$ (see §VIII.2.7 [17]).

The regular representation T of G' gives rise to a canonical regular H-projection-valued measure $\bar P$: $\bar P(W)f=Ch_Wf$, where $f\in H$, $W\in Bf(X)$, Ch_W is the characteristic function of W. Therefore, $T_h\bar P(W)=\bar P(h\circ W)T_h$ for each $h\in G'$ and $W\in Bf(X)$, since $\rho_\nu(h,h^{-1}\circ g)\rho_\nu(h,g)=1=\rho_\nu(e,g)$ for each $(h,g)\in G'\times X$, $Ch_W(h^{-1}\circ g)=Ch_{h\circ W}(g)$ and $T_h(\bar P(W)f(g))=\rho_\nu(h,g)^{1/2}\bar P(h\circ W)f(h^{-1}\circ g)$. Thus $<T,\bar P>$ is a system of imprimitivity for G' over X, which is denoted T^ν. This means that conditions $SI(i-iii)$ are satisfied:

$SI(i)$ T is a unitary representation of G';

$SI(ii)$ $\bar P$ is a regular H-projection-valued Borel measure on X and

$SI(iii)$ $T_h\bar P(W)=\bar P(h\circ W)T_h$ for all $h\in G'$ and $W\in Bf(X)$.

For each $F\in L^\infty(X,\nu,\mathbf{C})$ let $\bar\alpha_F$ be the operator in $\mathsf{L}(H)$ consisting of multiplication by F: $\bar\alpha_F(f)=Ff$ for each $f\in H$. The map $F\to\bar\alpha_F$ is an isometric $*$-isomorphism of $L^\infty(X,\nu,\mathbf{C})$ into $\mathsf{L}(H)$ (see §VIII.19.2[17]). Therefore, Propositions VIII.19.2,5[17] (using the approach of this particular case given above) are applicable in our situation.

If $\bar p$ is a projection onto a closed T^ν-stable subspace of H, then $\bar p$ commutes with all $\bar P(W)$. Hence $\bar p$ commutes with multiplication by all $F\in L^\infty(X,\nu,\mathbf{C})$, so by §VIII.19.2 [17] $\bar p=\bar P(V)$, where $V\in Bf(X)$. Also $\bar p$ commutes with all T_h, $h\in G'$, consequently, $(h\circ V)\setminus V$ and $(h^{-1}\circ V)\setminus V$ are ν-null for each $h\in G'$, hence $\nu((h\circ V)\bigtriangleup V)=0$ for all $h\in G'$. In view of ergodicity of ν and Proposition VIII.19.5 [17] either $\nu(V)=0$ or

$\nu(X \setminus V) = 0$, hence either $\bar{p} = 0$ or $\bar{p} = I$, where I is the unit operator. Hence T is the irreducible unitary representation.

5.2. Theorem. *On a loop or a diffeomorphism group G there exists a stochastic process, which generates a quasi-invariant measure μ relative to a dense subgroup G' such that the associated regular unitary representation $T^\mu : G' \to U(L^2(G, \mu, \mathbf{C}))$ is irreducible.*

Proof. From the construction of G' and μ in Theorem 3.3 it follows that, if a function $f \in L^1(G, \mu, \mathbf{C})$ satisfies the following condition $f^h(g) = f(g)$ $(mod\ \mu)$ by $g \in G$ for each $h \in G'$, then $f(x) = const$ $(mod\ \mu)$, where $f^h(g) := f(hg)$, $g \in G$.

Let $f(g) = Ch_U(g)$ be the characteristic function of a subset U, $U \subset G$, $U \in Af(G, \mu)$, then $f(hg) = 1 \Leftrightarrow g \in h^{-1}U$. If $f^h(g) = f(g)$ is true by $g \in G$ μ-almost everywhere, then $\mu(\{g \in G : f^h(g) \ne f(g)\}) = 0$, that is $\mu((h^{-1}U) \triangle U) = 0$, consequently, the measure μ satisfies the condition (P) from §VIII.19.5 [17], where $A \triangle B := (A \setminus B) \cup (B \setminus A)$ for each $A, B \subset G$. For each subset $E \subset G$ the outer measure is bounded, $\mu^*(E) \le 1$, since $\mu(G) = 1$ and μ is non-negative, consequently, there exists $F \in Bf(G)$ such that $F \supset E$ and $\mu(F) = \mu^*(E)$. This F may be interpreted as the representative of the least upper bound in $Bf(G)$ relative to the latter equality. In view of the Proposition VIII.19.5 [17] the measure μ is ergodic.

In view of Theorems 2.1.7 and 2.8 the Wiener process on the Hilbert manifold G induces the Wiener process on the Hilbert space T_eG with the help of the manifold exponential mapping. Then the left action L_h of G' on G induces the local left action of G' on a neighbourhood V of 0 in T_eG with $\nu(V) > 0$, where ν is induced by μ. A class of compact subsets approximates from below each measure μ^f, $\mu^f(dg) := |f(g)|\mu(dg)$, where $f \in L^2(G, \mu, \mathbf{C}) =: H$. Due to the Egorov Theorem II.1.11 [17] for each $\epsilon > 0$ and for each sequence $f_n(g)$ converging to $f(g)$ for μ-almost every $g \in G$, when $n \to \infty$, there exists a compact subset K in G such that $\mu(G \setminus \mathsf{K}) < \epsilon$ and $f_n(g)$ converges on K uniformly by $g \in \mathsf{K}$, when $n \to \infty$.

In view of Lemma IV.4.8 [48] the set of random variables $\{\phi(B_{t_1}, ..., B_{t_n}) : t_i \in [t_0, T], \phi \in C_0^\infty(\mathbf{R^n}), n \in \mathbf{N}\}$ is dense in $L^2(\mathsf{F}_T, \mu)$, where $T > t_0$. In accordance with Lemma IV.4.9 [48] the linear span of random variables of the type $\{exp\{\int_0^T h(t)dB_t(\omega) - \int_0^T h^2(t)dt/2\} : \quad h \in L^2[t_0, T]$ (deterministic) $\}$ is dense in $L^2(\mathsf{F}_T, \mu)$, where $T > t_0$. Therefore, in view of Girsanov Theorem 2.1.1 and Theorem 5.4.2 [54] the following space $sp_{\mathbf{C}}\{\psi(g) := (\rho(h, g))^{1/2} : h \in G'\} =: Q$ is dense in H, since $\rho_\mu(e, g) = 1$ for each $g \in G$ and $L_h : G \to G$

are diffeomorphisms of the manifold G, $L_h(g) = hg$. Finally we get from Theorem 3.3 above that there exists μ, which is ergodic and Conditions (i, ii) of Theorem 5.1.2 are satisfied. Evidently G' and G are infinite and dense in themselves. Hence from Theorem 5.1.2 the statement of this theorem, follows.

5.3. Note. Then analogously to §3.3 there can be constructed quasi-invariant and pseudo-differentiable measures on the manifold M relative to the action of the diffeomorphism group G_M such that $G' \subset G_M$. Then Poisson measures on configuration spaces associated with either G or M can be constructed and producing new unitary representations including irreducible as in [40].

Having a restriction of a transition measure μ from §3.3 on a proper open neighbourhood of e in G it is possible to construct a quasi-invariant σ-finite nonnegative measure m on G such that $m(G) = \infty$ using left shifts L_h on the paracompact G. Analogously such measure can be constructed on the manifold M in the case of the diffeomorphism group using Wiener processes on M. For definite μ in view of Theorems 2.9 [40] and 5.2 the corresponding Poisson measure P_m is ergodic. Therefore, Theorems 3.4, 3.6, 3.9, 3.10, 3.13 and 3.14 [40] also encompass the corresponding class of measures m and P_m arising from the constructed in §3.3 transition measures.

In view of Proposition II.1 [47] for the separable Hilbert space H the unitary group endowed with the strong operator topology $U(H)_s$ is the Polish group. Let $U(H)_n$ be the unitary group with the metric induced by the operator norm. In view of the Pickrell's theorem (see §II.2 [47]): if $\pi : U(H)_n \to U(V)_s$ is a continuous representation of $U(H)_n$ on the separable Hilbert space V, then π is also continuous as a homomorphism from $U(H)_s$ into $U(V)_s$. Therefore, if $T : G' \to U(H)_s$ is a continous representation, then there are new representations $\pi \circ T : G' \to U(V)_s$. On the other hand, the unitary representation theory of $U(H)_n$ is the same as that of $U_\infty(H) := U(H) \cap (1 + L_0(H))$, since the group $U_\infty(H)$ is dense in $U(H)_s$.

5.4. Remark. Let μ be a Borel regular Radon non-negative quasi-invariant measure on a topological Hausdorff group G relative to a dense subgroup G' with a continuous quasi-invariance factor $\rho_\mu(x, y)$ on $G' \times G$ and $0 < \mu(G) < \infty$. Suppose that $V : S \to U(H_V)$ is a strongly continuous unitary representation of a closed subgroup S in G'. There exists a Hilbert space $L^2(G, \mu, H_V)$ of equivalence classes of measurable functions $f : G \to$

H_V with a finite norm

$$(1)\ \|f\| := (\int_G \|f(g)\|^2_{H_V}\mu(dg))^{1/2} < \infty.$$

Then there exists a subspace Ψ_0 of functions $f \in L^2(G,\mu,H_V)$ such that $f(hy) = V_{h^{-1}}f(y)$ for each $y \in G$ and $h \in S$, the closure of Ψ_0 in $L^2(G,\mu,H_V)$ is denoted by $\Psi^{V,\mu}$. For each $f \in \Psi^{V,\mu}$ there is defined a function

$$(2)\ (T^{V,\mu}_x f)(y) := \rho^{1/2}_\mu(x,y)f(x^{-1}y),$$

where $\rho_\mu(x,y) := \mu_x(dy)/\mu(dy)$ is a quasi-invariance factor for each $x \in G'$ and $y \in G$, $\mu_x(A) := \mu(x^{-1}A)$ for each Borel subset A in G. Since $(T^{V,\mu}_x f)(hy) = V_{h^{-1}}((T_x f)(y))$, then $\Psi^{V,\mu}$ is a $T^{V,\mu}$-stable subspace. Therefore, $T^{V,\mu} : G' \to U(\Psi^{V,\mu})$ is a strongly continuous unitary representation, which is called induced and denoted by $Ind_{S\uparrow G'}(V)$.

5.5.1. Note. Let G be a topological Hausdorff group with a non-negative quasi-invariant measure μ relative to a dense subgroup G'. Suppose that there are two closed subgroups K and N in G such that $K' := K \cap G'$ and $N' = N \cap G'$ are dense subgroups in K and N respectively. We say that K and N act regularly in G, if there exists a sequence $\{Z_i : i = 0, 1, ...\}$ of Borel subsets Z_i satisfying two conditions:

(i) $\mu(Z_0) = 0$, $Z_i(k,n) = Z_i$ for each pair $(k,n) \in K \times N$ and each i;

(ii) if an orbit O relative to the action of $K \times N$ is not a subset of Z_0, then $\mathsf{O} = \bigcap_{Z_i \supset \mathsf{O}} Z_i$, where $g(k,n) := k^{-1}gn$. Let $T^{V,\mu}$ be a representation of G' induced by a unitary representation V of K' and a quasi-invariant measure μ (for example, as in §3). We denote by $T^{V,\mu}_{N'}$ a restriction of $T^{V,\mu}$ on N' and by D a space $K \setminus G/N$ of double coset classes KgN.

5.5.2. Theorem. *There are a unitary operator A on $\Psi^{V,\mu}$ and a measure ν on a space D such that*

$$(1)\ A^{-1}T^{V,\mu}_n A = \int_{\mathsf{D}} T_n(\xi)d\nu(\xi)$$

for each $n \in N'$. (2). *Each representation $N' \ni n \mapsto T_n(\xi)$ in the direct integral decomposition* (1) *is defined relative to the equivalence of a double coset class ξ. For a subgroup $N' \cap g^{-1}K'g$ its representations $\gamma \mapsto V_{g\gamma g^{-1}}$ are equivalent for each $g \in G'$ and representations $T_{N'}(\xi)$ can be taken up to their equivalence as induced by $\gamma \mapsto V_{g\gamma g^{-1}}$.*

Proof. A quotient mapping $\pi_{\mathsf{X}} : G \to G/K =: \mathsf{X}$ induces a measure $\hat{\mu}$ on X such that $\hat{\mu}(E) = \mu(\pi_{\mathsf{X}}^{-1}(E)) =: (\pi_{\mathsf{X}}^*\mu)(E)$ for each Borel subset E in X. In view of Radon-Nikodym theorem II.7.8 [17] for each $\xi \in \mathsf{D}$ there exists a measure μ_ξ on X such that

$$(3)\ d\hat{\mu}(x) = d\nu(\xi)d\mu_\xi(x),$$

where $x \in \mathsf{X}$, $\nu(E) := (s^*\mu)(E)$ for each Borel subset E in D, $s : G \to \mathsf{D}$ is a quotient mapping. In view of §26 [45] and Formula (3) the Hilbert space $H^V := L^2(\mathsf{X}, \hat{\mu}, H_V)$ has a decomposition into a direct integral

$$(4)\ H^V = \int_{\mathsf{D}} H(\xi)d\nu(\xi),$$

where H_V denotes a complex Hilbert space of the representation $V : K' \to U(H_V)$. Therefore,

$$\|f\|^2_{H^V} = \int_{\mathsf{D}} \|f\|^2_{H(\xi)}d\nu(\xi).$$

From Formulas (4) and 5.4.(1, 2) we get the first statement of this theorem for a subspace $\Psi^{V,\mu}$ of H^V.

If $f \in L^2(\mathsf{X}, \hat{\mu}, H_V)$, then $\pi_{\mathsf{X}}^* f := f \circ \pi_{\mathsf{X}} \in L^2(G, \mu, H_V)$. This induces an embedding π_{X}^* of H^V into $\Psi^{V,\mu}$. Let F be a filterbase of neighbourhoods A of K in G such that $A = \pi_{\mathsf{X}}^{-1}(S)$, where S is open in X, hence $0 < \mu(A) \le \mu(G)$ due to quasi-invaraince of μ on G relative to G'. Let $\psi \in \xi \in \mathsf{D}$, then $\psi = Kg_\xi$, where $g_\xi \in G$, hence $\psi = \psi(N \cap g_\xi^{-1}Kg_\xi)$. In view of Formula (3) for each $x \in N'$ and $\eta = Kx$ we get

$$(5)\quad \rho^{1/2}_{\mu_\xi}(\eta, \xi) = \lim_{\mathsf{F}}[\int_A \rho^{1/2}(x, zg_\xi)d\mu(z)/\mu(A)],$$

since by the supposition $\rho_\mu(h, y)$ is continuous on $G' \times G$ (see also §1.6 [16] and §§3.3, 5.1.1). Therefore,

$$(a, T_x(\xi)b)_{H^V} = \lim_{\mathsf{F}}[\int_A (\pi^*a, \rho^{1/2}_\mu(x, zg_\xi)(\pi^*b)^{zg_\xi}_x)_{\Psi^{V,\mu}}d\mu(z)/\mu(A)]$$

for each $x \in N'$ and $a, b \in H^V$, where $f^h_z(\zeta) := f(z^{-1}h\zeta)$ for a function f on G and $h, z, \zeta \in G$. In view of the cocycle condition $\rho_\mu(yx, z) = \rho_\mu(x, y^{-1}z)\rho_\mu(y, z)$ for each $x, y \in G'$ and $z \in G$ we get $T_{yx}(\xi) = T_y(\xi)T_x(\xi)$ for each $x, y \in N'$ and $T_x(\xi)$ are unitary representations of N'. Then

$$(a, T_{yx}(\xi)b)_{H^V} = \lim_{\mathsf{F}}[\int_A (\pi^*a, V_{g_\xi y g_\xi^{-1}}[\rho^{1/2}_\mu(x, zg_\xi)(\pi^*b)^{zg_\xi}_x])_{\Psi^{V,\mu}}d\mu(z)/\mu(A)]$$

for each $y \in N' \cap g_\xi^{-1} K' g_\xi$. Hence the representation $T_x(\xi)$ in the Hilbert space $H(\xi)$ is induced by a representation $(N' \cap g_\xi^{-1} K' g_\xi) \ni y \mapsto V_{g_\xi y g_\xi^{-1}}$.

5.6.1. Note. Let V and W be two unitary representations of K' and N' (see §5.5.1). In addition let K and N be regularly related in G and $V \hat{\otimes} W$ denotes an external tensor product of representations for a direct product group $K \times N$. In view of §5.4 a representation $T^{V,\mu} \hat{\otimes} T^{W,\mu}$ of an external product group $\mathsf{G} := G \times G$ is equivalent with an induced representation $T^{V \hat{\otimes} W, \mu \otimes \mu}$, where $\mu \otimes \mu$ is a product measure on G. A restriction of $T^{V \hat{\otimes} W, \mu \otimes \mu}$ on $\tilde{G} := \{(g,g) : g \in G\}$ is equivalent with an internal tensor product $T^{V,\mu} \otimes T^{W,\mu}$.

5.6.2. Theorem. *There exists a unitary operator A on $\Psi^{V \hat{\otimes} W, \mu \otimes \mu}$ such that*

$$(1)\ A^{-1} T^{V,\mu} \otimes T^{W,\mu} A = \int_{\mathsf{D}} T(\xi) d\nu(\xi),$$

where ν is an admissible measure on a space $\mathsf{D} := N \setminus G / K$ of double cosets.

(2). *Each representation $G' \ni g \mapsto T_g(\xi)$ in Formula* (1) *is defined up to the equivalence of ξ in D. Moreover, $T(\xi)$ is unitarily equivalent with $T^{\tilde{V} \otimes \tilde{W}, \mu \otimes \mu}$, where $\tilde{V}$ and $\tilde{W}$ are restrictions of the corresponding representations $y \mapsto V_{gyg^{-1}}$ and $y \mapsto W_{\gamma y \gamma^{-1}}$ on $g^{-1} K' g \cap \gamma^{-1} N' \gamma$, $g, \gamma \in G'$, $g\gamma^{-1} \in \xi$.*

Proof. In view of §18.2 [6] $P \setminus \mathsf{G} / \tilde{G}$ and $K \setminus G / N$ are isomorphic Borel spaces, where $P = K \times N$. In view of Theorem 5.5.2 there exists a unitary operator A on a subspace $\Psi^{V \hat{\otimes} W, \mu \otimes \mu}$ of the Hilbert space $L^2(\mathsf{G}, \mu \otimes \mu, H_V \otimes H_W)$ such that

$$A^{-1} T^{V \hat{\otimes} W, \mu \otimes \mu}|_{\tilde{G}} A = \int_{\mathsf{D}} T_{\tilde{G}}(\xi) d\nu(\xi),$$

where each $T_{\tilde{G}}(\xi)$ is induced by a representation $(y,y) \mapsto (V \hat{\otimes} W)_{(g,\gamma)(y,y)(g,\gamma)^{-1}}$ of a subgroup $\tilde{G}' \cap (g,\gamma)^{-1} (K \times N)(g,\gamma)$, the latter group is isomorphic with $S := g^{-1} K' g \cap \gamma^{-1} N' \gamma$, that gives a representation $\tilde{V} \hat{\otimes} \tilde{W}$ of a subgroup $S \times S$ in G. Therefore, we get a representation $T^{\tilde{V} \hat{\otimes} \tilde{W}, \mu \otimes \mu}$ equivalent with $Ind_{(S \times S) \uparrow \mathsf{G}'} (\tilde{V} \hat{\otimes} \tilde{W})|_{\tilde{G}'}$.

5.7. Note. Formulas $(3-5)$ from §5.5.2 also show how a measure ν on a groupoid $Y^\xi(M, s_0; N, y_0)$ induces a measure μ on $(S^M N)_\xi$ and produces an expression for a quasi-invariance factor on a loop monoid and then on a loop group.

6 Appendix.

Let us remind the principles of the Wiener processes on manifolds.

Let $\bar{G}$ be a complete separable relative to its metric $\bar{\rho}$ C^∞-manifold on a Banach space $\bar{Y}$ over $\mathbf{R}$ such that it contains a dense C^∞-submanifold G on a Hilbert space Y over $\mathbf{R}$, where G is also separable and complete relative to its metric ρ. Let $\tau_G : TG \to G$ be a tangent bundle on G. Let $\theta : Z_G \to G$ be a trivial bundle on G with the fibre Z such that $Z_G = Z \times G$, then $L_{1,2}(\theta, \tau_G)$ be an operator bundle with a fibre $L_{1,2}(Z, Y)$, where $Z, Z_1, ..., Z_n$ are Hilbert spaces, $L_{n,2}(Z_1, ..., Z_n; Z)$ is a subspace of a space of all Hilbert-Schmidt n times multilinear operators from $Z_1 \times ... \times Z_n$ into Z. Then $L_{n,2}(Z_1, ..., Z_n; Z)$ has the structure of the Hilbert space with the scalar product denoted by

$$\sigma_2(\phi, \psi) := \sum_{j_1,...,j_n=1}^{\infty} (\phi(e^{(1)}_{j_1}, ..., e^{(n)}_{j_n}), \psi(e^{(1)}_{j_1}, ..., e^{(n)}_{j_n}))$$

for each pair of its elements ϕ, ψ. It does not depend on a choise of the orthonormal bases $\{e(k)_j : j\}$ in Z_k. Let $\Pi := \tau_G \oplus L_{1,2}(\theta, \tau_G)$ be a Whitney sum of bundles τ and $L_{1,2}(\theta, \tau_G)$. If (U_j, ϕ_j) and (U_l, ϕ_l) are two charts of G with an open non-void intersection $U_j \cap U_l$, then to a connecting mapping $f_{\phi_l,\phi_j} = \phi_l \circ \phi_j^{-1}$ there corresponds a connecting mapping $f_{\phi_l,\phi_j} \times f'_{\phi_l,\phi_j}$ for the bundle Π and its charts $U_j \times (Y \oplus L_{1,2}(Z, Y))$ for $j = 1$ or $j = 2$, where f' denotes the strong derivative of f, $f'_{\phi_l,\phi_j} : (a^{\phi_j}, A^{\phi_j}) \mapsto (f'_{\phi_l,\phi_j} a^{\phi_j}, f'_{\phi_l,\phi_j} \circ A^{\phi_j})$, $a^\phi \in Y$ and $A^\phi \in L_{1,2}(Z, Y)$ for the chart (U, ϕ), $f'_{\phi_l,\phi_j} \circ A^{\phi_j} := f'_{\phi_l,\phi_j} A^{\phi_j} f'^{-1}_{\phi_l,\phi_j}$. Such bundles are called quadratic. Then there exists a new bundle J on G with the same fibre as for Π, but with new connecting mappings: $J(f_{\phi_l,\phi_j}) : (a^{\phi_j}, A^{\phi_j}) \mapsto (f'_{\phi_l,\phi_j} a^{\phi_j} + tr(f"_{\phi_l,\phi_j}(A^{\phi_j}, A^{\phi_j}))/2, f'_{\phi_l,\phi_j} \circ A^{\phi_j})$, where $tr(A)$ denotes a trace of an operator A. Then using sheafs one gets the Itô functor $I : I(G) \to G$ from the category of manifolds to the category of quadratic bundles.

On a Hilbert space W a distribution $\gamma_{b,B}$ is called Gaussian, if its Fourier transform is the following:

$$F'(\gamma_{b,B})(v) = exp\{-(Bv, v)/2 + i(b, v)\},$$

where B is the corresponding symmetric bounded nonnegative nondegenerate nuclear operator on W, $b \in W$, $v \in W$.

On Y let B be a nuclear selfadjoint linear nonnegative operator with $ker(B) = \{0\}$, then for each $t > 0$ it defines a Gaussian measure μ_{tB} with zero mean and correlation operator tB. It is defined with the help of the Hilbert-Schmidt structure in Y (that is, the rigged Hilbert space): $Y = Y'_-$, $Y'_+ \hookrightarrow Y_0$ and $Y_0 \hookrightarrow Y'_-$ are Hilbert-Schmidt embeddings $B^{1/2}$, $Y' := Y'_+$, where $(x, y)_+ = (B^{-1}x, B^{-1}y)$ is the scalar product in Y'_+ induced from the dense subspace $B^{-1}Y'_-$, $(x, y)_0 = (B^{-1/2}x, B^{-1/2}y)$ is the scalar product in Y_0 induced from the dense subspace $B^{-1/2}Y'_-$, where (x, y) is the scalar product in $Y = Y'_-$ for each $x, y \in Y$. By the definition a Wiener process $w(t, \omega)$ for $0 \le t_0 \le t < \infty$ with values in Y is a stochastic process for which

(1) the differences $w(t_4, \omega) - w(t_3, \omega)$ and $w(t_2, \omega) - w(t_1, \omega)$ are independent for each $t_0 \le t_1 < t_2 \le t_3 < t_4$;

(2) the random variable $w(t + \tau, \omega) - w(t, \omega)$ has a distribution $\mu_{\tau B}$, where $w(t_0, \omega) := 0$, (Ω, F, P) is a probability space of a set Ω with a σ-algebra F of its subset and a probability measure P.

Then consider the class $\mathsf{K}(Y)$ of stochastic processes $B(t, \omega)$ with values in $L_{1,2}(\mathsf{H}, Y)$ and satisfying $\beta^2(B) = \int_{t_0}^{\tau} \mathsf{M}\sigma_2(B(t, \omega), B(t, \omega))dt < \infty$, the space of all such operators is denoted by $L_2(\Omega, Y)$, where M denotes the operation of the mean value, the embedding of H into Y is a Hilbert-Schmidt operator, $\omega \in \Omega$, (Ω, F, P):

(3) for each $t \ge t_0$ the quantity $B(t, \omega)$ is F_t-measurable, where F_t is a flow of σ-algebras, that is, a monotone set of σ-algebras ($\mathsf{F}_t \subset \mathsf{F}_s$ for each $s \ge t \ge t_0$) such that for each $s \le t$ the random variable $w(s, \omega)$ is F_t-measurable, $w(\tau, \omega) - w(s, \omega)$ is independent from F_t for each $\tau > s \ge t$.

Let $\mathsf{K}_0(Y)$ be the subset of $\mathsf{K}(Y)$ consisting of step functions $B(t, \omega) = B_j(\omega)$ for each $t_j \le t < t_{j+1}$, where $t_0 < t_1 < ... < t_n = \tau$ is a partition of the segment $[t_0, \tau]$ in $\mathbf{R}$. In $\mathsf{K}_0(Y)$ the Itô stochastic integral is defined by $\mathsf{I}(B) = \int_{t_0}^{\tau} B(t, \omega)dt = \sum_{j=0}^{n-1} B_j(\omega)[w(t_{j+1}, \omega) - w(t_j, \omega)]$. It has the extension $\mathsf{I} : \mathsf{K}(Y) \to L_2(\Omega, Y)$. Let $a(t, \omega)$ be an F_t-measurable function with values in Y such that $\int_{t_0}^{\tau} \mathsf{M}\|a(t, \omega)\|^2 dt < \infty$ and let $\xi_0(\omega)$ be an F_{t_0}-measurable random variable. A stochastic process of the type

$$\xi(t, \omega) = \xi_0(\omega) + \int_{t_0}^{t} a(s, \omega)ds + \int_{t_0}^{t} B(s, \omega)dw(s, \omega)$$

is said to have a stochastic differential and it is written as follows:

$$d\xi = a(t, \omega)dt + B(t, \omega)dw(t, \omega).$$

If $f(t,x)$ is continuously differentaible by t and continuously twice strongly differentiable by x function from $[t_0,\tau] \times Y$ into Y and they are bounded, then

$$f(t,\xi(t,\omega)) = f(t_0,\xi_0(\omega)) + \int_{t_0}^{t} \{f'_s(s,\xi(s,\omega)) + f'_x(s,\xi(s))a(s,\omega)+$$

$$tr(B^*(s,\omega)f"_{x,x}(s,\xi(s,\omega))B(s,\omega))/2\}ds + \int_{t_0}^{t} f'_x(s,\xi(s,\omega))B(s,\omega)dw(s,\omega)$$

in accordance with the Itô's formula.

Let the manifold G be supplied with the connection. A curve $c : [-2,2] \to G$ is called a geodesic if $\nabla \dot{c}(t)/dt = 0$. In view of Corollary 1.6.8 [26] there exists an open neighbourhood $\tilde{T}G$ of the submanifold G of TG such that for every $X \in \tilde{T}G$ the geodesic $c_X(t)$ is defined for $|t| < 2$, where TG denotes the tangent bundle. The exponential mapping $exp^G : \tilde{T}G \to G$ is defined by the formula $X \mapsto c_X(1)$. The restriction $exp^G|_{\tilde{T}G \cap T_pG}$ will also be denoted by exp^G_p. Then there is defined the mapping $I(exp^G) : I(\tilde{T}G) \to I(G)$ such that for each chart (U,ϕ) the mapping $I(exp^\phi) : Y \oplus L_{1,2}(Z,Y) \to Y \oplus L_{1,2}(Z,Y)$ is given by the following formula:

$$I(exp^\phi)(a^\phi, A^\phi) = (a^\phi - \quad tr(\Gamma^\phi(A^\phi, A^\phi))/2, A^\phi),$$

where Γ denotes the Christoffel symbol.

Therefore, if $\mathsf{R}_{x,0}(a,A)$ is a germ of diffusion processes at a point $y = 0$ of the tangent space T_xG, then $\tilde{exp}_x \mathsf{R}_{x,0}(a,A) := \mathsf{R}_x(I(exp_x)(a,A))$ is a germ of stochastic processes at a point x of the manifold G. The germs $\tilde{exp}_x \mathsf{R}_{x,0}(a,A)$ are called stochastic differentials and the Itô bundle is called the bundle of stochastic differentials such that $\mathsf{R}_{x,0}(a,A) =: a_x dt + A_x dw$. A section U of the vector bundle $\Pi = \tau_Y \oplus L_{1,2}(\theta,\tau_Y)$ is called the Itô field on the manifold G and it defines a field of stochastic differentials $\mathsf{R}_x(I(exp_x)(a,A)) = \tilde{exp}_x(a_x dt + A_x dw)$. A random process ξ has a stochastic differential defined by the Itô field $\mathsf{U} : d\xi(s,\omega) = \tilde{exp}_{\xi(s,\omega)} \mathsf{R}(a_{\xi(s,\omega)}, A_{\xi(s,\omega)})$ if the following conditions are satisfied: for $\nu_{\xi(s)}$-almost every $x \in Y$ there exists a neighbourhood V_x of a point x and a diffusion process $\eta_x(t,\omega)$ belonging to the germ $\mathsf{R}_x(I(exp_x))(a,A)$ such that $P_{s,x}\{\xi(t,\omega) = \eta_x(t,\omega) : \xi(t,\omega) \in V_x, t \ge s\} = 1$ $\nu_{\xi(s)}$-almost everywhere, where $P_{s,x}(S) := P\{S : \xi(s,\omega) = x\}$, S is a P-measurable subset of Ω, $\nu_{\xi(s)}(F) := P\{\omega : \xi(s,\omega) \in F\}$ (see Chapter 4 in [7]).

If $\mathsf{U}(t) = (a(t), A(t))$ is a time dependent Itô field, then a random process $\xi(t,\omega)$ having for each $t \in [0,T]$ a stochastic differential $d\xi = exp_{\xi(t,\omega)}(a_{\xi(t,\omega)}dt + A_{\xi(t,\omega)}dw)$ is called a stochastic differential equation on the manifold G, the process $\xi(t,\omega)$ is called its solution (see Chapter VII in [10]). As usually a flow of σ-algebras consistent with the Wiener process $w(t,\omega)$ is a monotone set of σ-algebras F_t such that $w(s,\omega)$ is F_t-measurable for each $0 \le s \le t$ and $w(\tau,\omega) - w(s,\omega)$ is independent from F_t for each $\tau > s \ge t$, where $\mathsf{F}_s \supset \mathsf{F}_t$ for each $0 \le t \le s$. If G is the manifold with the uniform atlas (see §2.1), the Itô field (a, A) and Christoffel symbols are bounded, then there exists the unique up to stochastic equivalence random evolution family $S(t,\tau)$ consistent with the flow of σ-algebras F_t generated by the solution $\xi(t,\omega)$ of the stochastic differential equation $d\xi = exp_{\xi(t,\omega)}(a_{\xi(t,\omega)}dt + A_{\xi(t,\omega)}dw)$ on G, that is, $\xi(\tau,\omega) = x$, $\xi(t,\omega) = S(t,\tau,\omega)x$ for each $t_0 \le \tau < t < \infty$ (see Theorem 4.2.1 [7]).

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